A Walrasian Theory of Sovereign Debt Auctions with Asymmetric Information*

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February, 2018

Abstract

Sovereign bonds are highly divisible, usually of uncertain quality, and auctioned in large lots to a large number of potential investors. This leads us to assume that no individual bidder can affect the bond price, and to develop a tractable Walrasian theory of Treasury auctions in which investors are asymmetrically informed about the quality of the bond. We characterize the price of the bond for different degrees of asymmetric information, both under discriminatory-price (DP) and uniform-price (UP) protocols. We endogenize information acquisition and show that DP protocols display multiple equilibria and are more likely to induce asymmetric information than UP protocols. This result has welfare implications as asymmetric information negatively affects the level, dispersion and volatility of sovereign bond prices.

*We thank Jakub Kastl, Felix Kubler, George Mailath, Aviv Nevo, Andy Postlewaite, Laura Veldkamp and seminar participants at Central Bank of Chile, EIEF, EUI, UBC, UPenn, Wharton, Zurich, the 2017 Cowles Conference on General Equilibrium at Yale and the 2017 SED Meetings in Edinburgh for comments. The usual waiver of liability applies.

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1 Introduction

Sovereign debt auctions play a critical role in allowing governments to finance their deficits. They have also been intimately involved in many major economic events, like the recent European crises. Figure 1 illustrates the challenges to modeling these auctions by displaying interest rate data for the 91-day Cetes from the Mexican government’s weekly auctions. These bonds were domestically denominated, sold in very small denominations and large lots, to a wide variety of investors, using auctions that alternated between uniform and discriminating price protocols.\(^1\) During the long period displayed in the figure, annual interest rate on these bonds fluctuated widely in response to different economic events, such as the Latin American debt crisis of the 1980s and the “Tequila Crisis” in 1995. It also includes a wide range of shocks, some public, some private, some learnable, some not. Some of these shocks affected common factors, like the probability of default, some of those shocks may have affected the private valuation of the government’s bonds, like liquidity shocks.

When we look at emerging market interest rate spreads (against LIBOR) more generally and include secondary market prices, we see that these spreads vary substantially across both countries and time, suggesting that uncertainty about the likelihood of default or renegotiation is an important component in sovereign debt markets. These spreads are partially accounted for by country fundamentals, like debt-to-output, and the growth rate of output, but also depend on common factors which are themselves partially accounted for by financial factors, like measures of risk pricing and uncertainty premia. In addition the high (on average) realized returns on emerging market debt, along with the great frequency of high “crisis” spreads relative to actual defaults suggest that the returns on these bonds include a substantial risk premium. See Aguiar et al. (2016b).

Sovereign debt auctions generally share the key features of Mexico’s Cetes: a large volume of the debt (in the form of sovereign bonds) is usually sold at one time to a large number of investors,\(^2\) and investors are free to try and buy as many units of bonds as they

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\(^1\)Cetes are zero-coupon bonds which investors can invest in directly using Cetesdirecto since 2010. Cetes remain among the most important public debt instruments in Mexico. In 2001, for example, Cetes alone represented 25% of all government securities, and were auctioned 180 times to 3,581 participating bidders.

\(^2\)Malvey, Archibald, and Flynn (1995) report that the U.S. Treasury typically receives 75-85 competitive
can afford. To analyze these auctions, we propose a novel model of auctions with three characteristics: (i) the good being auctioned is perfectly divisible, (ii) the number of bidders is large, and (iii) there is both common uncertainty about the good quality and about the mass of investors who participate in the auction. Given these three characteristics, the price-quantity strategic aspects of standard auction theory become less relevant, and a price-taking, or Walrasian, analysis naturally emerges as a good approximation. We will show that Walrasian auctions are particularly tractable and allow for an analysis of the role of information on equilibrium prices and, once we include information acquisition, the role of each auction protocol in determining the level of asymmetric information.

Sovereign debt auctions are generally conducted using one of two formats - uniform-price auctions and discriminating-price auctions - with discriminating-price auctions being slightly more prevalent and uniform-price auctions being the standard method in the United States. We therefore consider both types of auction protocols and examine their

3There are papers in the auction literature that yield price-taking as the number of bidders get large. A recent example is Fudenberg, Mobius, and Szeidl (2007) show that the equilibria of large double auctions with correlated private values are essentially fully revealing and approximate price-taking behavior. Another is Reny and Perry (2006) who show a similar result when bidders have affiliated values and prices are on a fine grid.

4The heterogeneity of treasury auction formats is well-documented. For example, the U.S. switched to a uniform-price format from a discriminating-price format in the 1970s, while Canada and Germany use the price-discriminating format. Bartolini and Cottarelli (2001) study a sample in which 39 out of 42 countries use discriminatory price auctions. Brenner, Galai, and Sade (2009) analyze a sample of 48 countries, out of
A key aspect of our model is information heterogeneity. We are able to characterize the equilibria of our auction model under both auction formats in the presence of this heterogeneity, and we develop a numerical example to illustrate its role. This success comes despite the fact that we deviate from the standard CARA preferences with normal shocks, and occurs in cases in which there is not perfect ex post information revelation.\(^5\)

Modeling information acquisition in sovereign debt markets allows us to uncover a new source of multiplicity. We show that discriminatory-price auctions have the potential to generate multiple equilibrium with different degrees of asymmetric information about the sovereign bonds quality. Interestingly, uniform-price auctions display a unique equilibrium level of asymmetric information. While the literature has considered other source of multiplicity in sovereign debt markets, stemming from self-fulfilling beliefs on the part of investors, it has neglected modeling how information acquisition and inference play a role in determining bond spreads.\(^6\)

Our paper fills an important gap in the sovereign debt literature which has typically focused on the strategic decision of the government; something we completely neglect.\(^7\) Most of the literature on sovereign debt has typically taken investors to be risk neutral and required that the return, adjusted for the probability of default, equals the risk-free rate. This means that the literature has neglected information acquisition and its aggregation in prices. Moreover, while there has been some attention to the timing of decisions in bond markets and the impact of debt maturity (see Aguiar et al. (2016a)), the actual mechanics of sovereign bond auctions and their impact on prices has been ignored. Our paper focuses on the neglected role of information acquisition when the auction is explicit which 24 use discriminatory-price auctions, 9 use uniform-price auctions and the rest use either both or an hybrid between the two.

\(^5\)It is well known that proving the existence of an equilibrium when prices are not fully revealing is very difficult (see Allen and Jordan (1998)), and that the CARA-normal case is special because it yields a linear price function.

\(^6\)There are two main approaches to the role of self-fulfilling beliefs in this literature; the first is Calvo (1988) (with successor papers Lorenzoni and Werning (2013) or Ayres et al. (2016)) and the second is Cole and Kehoe (2000) (with successor papers Aguiar et al. (2015)) and Aguiar et al. (2017).

\(^7\)See for example Eaton and Gersovitz (1981), the review articles by Aguiar and Amador (2013) and Aguiar et al. (2016b), and the recent quantitative literature by Aguiar and Gopinath (2006), Arellano (2008), Chatterjee and Eyigungor (2012), Bocola and Dovis (2016).
itly modeled.

There has been a recent effort to empirically document the implication of the different auction protocols and of the information sharing across dealers on the revenue of governments. For the former, see the survey by Hortacşu (2011). For the latter, see Boyarchenko, Lucca, and Veldkamp (2017). As our setting is very different, we discuss our paper’s relationship with this literature once we have presented our model. There we also discuss the relationship to general equilibrium theory and to the auctions literature more broadly.

2 Model with Exogenous Information Asymmetry

2.1 Environment

This is a two-period model featuring a measure one of ex ante identical risk averse potential investors and a government. The government is modeled mechanically: it needs to raise $D$ units of the numeraire good in period one by auctioning a bond that promises repayment in period two. This bond is risky because it constitutes a claim to one real unit in period two only if the government does not default. If the government defaults, then investors cannot recover any of the investment. The probability of default, $\kappa$, is random and takes on two values, $\kappa_g < \kappa_b$. The ex-ante probability of each value is given by $f(g)$ and $f(b)$ respectively, with $f(g) + f(b) = 1$. Since the default probability determines the expected repayment of the bond, we refer to the realization of $\kappa$ as the quality shock.

It is straightforward to think of the $\kappa_\theta$ realizations being themselves governed by an aggregate public shock $\nu$. This aggregate public shock can be thought of as being drawn at the beginning of the period, and the two possible realizations of the quality shock, $\kappa_\theta(\nu)$, are determined by its realization. Because adding a public shock leaves the analysis unchanged, we suppress consideration of $\nu$, however the analysis that follows can be thought of as conditional on a given realization of $\nu$.

The government sells these bonds in an auction in period one. If the amount of money raised at auction falls short of $D$, then we assume that the government simply defaults on

\footnote{The supply of bonds being auction is therefore random and depends on the realized bids. This is meant to capture the impact of revenue needs on the government’s auction behavior.}
any bonds that it sold in period one (we can take this to also mean that they defaulted on the bonds coming due in period one).

The objective of investors is to maximize their expected, strictly concave, flow utility functions $U$ over their second period consumption. Each investor has wealth $W$ in period one and can either invest in a risk-free bond (storage) or the risky bond being auctioned by the government. In addition to the quality shock that determines the probability of default, there is a demand shock, which we model as a random share of investors who show up to the government’s auction. We denote the random fraction of the potential investors who make it to the auction by $\eta$. Those that do not make it to the auction have no choice but to invest all of their wealth in the risk-free bond and eat the proceeds in the second period. The investors who do make it to the auction have the option to bid and invest a fraction of their wealth in the risky government bond, with the remainder invested in storage.

We assume that $\eta$ is continuously distributed on the interval $[0, \eta_M]$ according to a continuous density function $g(\eta)$ that is nonzero everywhere on the interior of the interval, with $\eta_M < 1$. We will denote the set of possible values of $\eta$ by $\mathcal{H}$, and refer to $s = (\theta, \eta)$ as the state of the world. Here $\theta \in \{g, b\}$ and $\kappa = \kappa_g$ if $\theta = g$, and $\kappa = \kappa_b$ if $\theta = b$. The set of states is denoted by $S = \{g, b\} \times \mathcal{H}$.

Two natural interpretations of $\eta$ are: (i) it governs the fraction of investors who suffer a liquidity shock and end up with wealth 0 and hence have nothing to invest, and (ii) it governs the fraction of investors who have a favorable outside investment opportunity and therefore choose not to bid on the risky bond at the auction. Viewed in this way, the shock to demand coming through $\eta$ can be thought of as correlated private value shocks in the context of the auction literature. The other shocks, the explicit shock $\theta$ and implicit shock $\nu$, are common value shocks in the language of the auction literature. $\nu$ is a public common value shock, while $\theta$ is a common value shock about which there is heterogeneous information.

At the auction, investors can submit multiple bids. Each bid is a price and quantity

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9This shock to the demand for the bond can be also interpreted as a shock to its supply, or the amount of funds that the government needs to raise at the auction in period one, $D/(1 - \eta)$. 

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pair \( \{P, B\} \) representing a commitment to purchase \( B \) units of the bond either at price \( P \) should the government decide to execute the bid. The government treats each bid independently, sorts all bids from the highest to the lowest bid price, and accepts all bids in descending order to the highest bid price at which the amount \( D \) is raised. We refer to this highest possible “lowest” accepted price as the marginal price \( \bar{P} \), and to bids above the marginal price as bids in the money.

The price that an investor has to pay when a bid is accepted depends on the auction protocol. We consider two protocols that are widely used in large volume auctions of a common good. In the first, the government sells bonds using a discriminatory-price (DP) auction (bonds are sold at the bid price, or “pay as you bid”). In the second, the government sells bonds using a uniform-price (UP) auction (all accepted bids are executed at the lowest accepted, or marginal, price). We assume that the government and the investors take the auction protocol as given. If the marginal price does not exactly clear the market (i.e. it generates revenue greater than \( D \)), then only a fraction of the bids at the marginal price are accepted and bonds are rationed pro-rata among investors. As we will show, rationing does not happen in equilibrium.

There will be two types of investors at the auction: those who are informed about \( \theta \) and those who are not. We denote by \( i \in \{I, U\} \) the type of investor and use \( n \in [0, 1] \) to denote the share of investors who are informed (\( I \)), with \( 1 - n \) denoting the share who are uninformed (\( U \)). Because informed (uninformed) investors are otherwise identical, they behave the same and we can refer to a representative informed (uninformed) investor. No investor is informed about \( \eta \), which means that all investors face uncertainty about the minimum price at which they can buy the bond conditional on their information (or lack thereof) about \( \theta \). Consistent with our mechanical modeling of the government we assume that it observes neither \( \theta \) nor \( \eta \) before the auction.

Investors lack commitment in two important dimensions. First, they cannot commit to honor any intertemporal contracts. We will take this to mean that they cannot borrow at the risk-free rate, nor can they make negative bids at the auction. Investors must therefore bid nonnegative quantities (\( B \geq 0 \)) and can spend no more than their wealth \( W \) on bonds. Second, they cannot commit to credibly share their information about \( \theta \). We will take this
to mean that there is no market for information about $\theta$.

A unit of the bond is a claim to a real unit of the numeraire good in period two. As this claim either pays 1 or 0, the range of possible prices is $P \in [0,1]$. Since investors will typically find it optimal to submit multiple bids, we start by taking the investors’ strategy to be a bid function $B^I(P|\theta)$ for the informed and $B^U(P)$ for the uninformed where $B : [0, 1] \rightarrow [0, W]$.

If $\bar{P}(s)$ is the marginal price in state $s$, then the amount that the government raises in this state in a UP auction is

$$\bar{P}(s)(1 - \eta) \int_{P(s)}^{1} \left[ nB^I(P|\theta(s)) + (1 - n)B^U(P) \right] dP,$$

where $\theta(s)$ is the quality shock in state $s$. The left-hand expression is simply the marginal price $\bar{P}(s)$ multiplied by the accepted number of bids given this marginal price. The amount the government raises is increasing in the marginal price $\bar{P}(s)$, but the number of accepted bids is decreasing in the marginal price. As market clearing equalizes this amount to $D$ there may be multiple price points at which the government raises the necessary amount $D$. The auction protocol we use is to always use the highest such price. This implies that the revenue raised by the government in the auction is declining in the marginal price.

The amount raised in a DP auction in state $s$ given marginal price $\bar{P}(s)$ is

$$(1 - \eta) \int_{P(s)}^{1} \left[ nB^I(P|\theta(s)) + (1 - n)B^U(P) \right] PdP.$$

In contrast to the UP auction, this amount is always declining in the marginal price $\bar{P}(s)$ since the prices are fixed at the bidding price while the number of accepted bids is decreasing in the marginal price. Market clearing equalizes this amount to $D$.

We assume that investors have rational expectations: the set of marginal prices, their probabilities and the states associated with them are all common knowledge before submitting the bids. After the auction has been performed and the realization of the marginal price has been revealed, informed and uninformed investors can make inferences with re-
spect to the state. For the informed investor this is straightforward since they know $\theta$ and can infer $\eta$ by inverting the price schedule. For the uninformed this is somewhat more complicated. If the price $\bar{P}(s)$ is uniquely generated by a quality shock, then they too can infer the state. If there exist two states $(g, \eta_g)$ and $(b, \eta_b)$ that have a common price, then they will still be able to update their beliefs about the set of possible states and their probabilities from observing the price. However, this ex-post information is of limited use since all of the investors must choose their bids prior to observing the price.

At the time they make their bids, the informed investors know $\theta$ and the probability distribution over $\eta$, while the uninformed investors know the probabilities distributions over $\theta$ and $\eta$. They can also compute the bidding strategies of the other investors so they know how the realized state will determine the marginal price at the auction. With this information they can make inferences about the set of states and their probabilities in the event they are able (or not able) to buy at a bid price of $P$.

In the DP auction it is a strictly dominating strategy to bid only at the possible marginal prices $\bar{P}(s)$. Otherwise, if the bid is made at a price slightly above $\bar{P}(s)$ the same bid is accepted but the investor pays a higher price. In the UP auction it is a weakly dominating strategy to do so. In light of this, we restrict our agents to only bid at marginal prices. With this restriction, we no longer need to think of our agents having a bidding strategy. We can instead think of them as choosing how many bonds to bid for at the marginal price for each state. Since bids only happen at marginal prices, we drop the notation $\bar{P}$ and just refer to $P$. Also, for this reason, we switch to a simpler and starker specification of prices and bids.

**Definition 1.** For each state $s = (\theta, \eta, \in S$, the marginal price is denoted $P(\theta, \eta)$ and the set of marginal prices by $\mathcal{P}$. An action for the uninformed investors is a function $B_U(\theta, \eta)$ which denotes the number of units bid at the marginal price $P(\theta, \eta)$. An action for the informed investors is a function $B_I(\theta, \eta | \hat{\theta})$ which denotes their bids at the various possible states when the realized $\theta$ is $\hat{\theta}$.

**Remark 1.** For any pair of states $(g, \eta_g)$ and $(b, \eta_b)$ s.t. $P(g, \eta_g) = P(b, \eta_b)$, bids at these states are perfect substitutes since they will be accepted and rejected in identical circumstances across realized states. Thus, the investor bids the total quantity $B(g, \eta_g) + B(b, \eta_b)$ at the price $P =
\[ P(g, \eta_g) = P(b, \eta_b). \] In this case, the uninformed investors will not be able to infer the state ex-post for this price \( P \).

**Remark 2.** This stark specification allows us to directly compare our auction to a competitive equilibrium. It is also particularly helpful when the set of \( \eta \)'s is finite, and so is the set of possible marginal prices. In our original specification of actions as bids on the set of all potential prices \( P \in [0, 1] \) this would mean that the bid function would be positive only at a finite set of points corresponding to those marginal prices. But even when \( \eta \) is continuous, the set of marginal prices is a strict subset of the set of potential prices.

### 3 Auction Equilibrium

We start with the problem of the uninformed investor. If the government ends up defaulting in the second period, the uninformed investor simply consumes the unit payoff from his risk-free bonds, which we denote by \( B_{RF}^U(s) \). If the government does not default, then the investor also consumes the unit payoff from his total purchases of the risky bond, which we denote by \( B_{R}^U(s) \). The expected payoff to an uninformed investor is given by

\[
\sum_{\theta \in \{g,b\}} \int_{\eta} \left\{ U(B_{RF}^U([\theta, \eta])) \kappa + U(B_{RF}^U([\theta, \eta]) + B_{R}^U([\theta, \eta])) (1 - \kappa) \right\} f(\theta) g(\eta) d\eta
\]

where we are summing over the conditional payoffs in each possible state \((\theta, \eta)\), weighted by the probability of that state. The total risky bonds purchased by an uninformed bidder in each state, \( B_{R}^U(s) \), is

\[
B_{R}^U(s) = \sum_{s': P(s') \geq P(s)} B^U(s').
\]
The total expenditures on these risky bonds, which determines the amount invested in the risk-free bond, depends upon the auction protocol and is given by:

\[ B_{RF}^U(s) = W - \left[ \sum_{s': P(s') \geq P(s)} B^U(s') P(s) \right], \quad (3) \]

**UP auction**

\[ B_{RF}^U(s) = W - \left[ \sum_{s': P(s') \geq P(s)} B^U(s') P(s) \right]. \quad (4) \]

**DP auction**

The investor cannot short-sell or borrow, so he faces a nonegativity constraint

\[ B^U(s) \geq 0 \text{ and } B_{RF}^U(s) \geq 0 \quad \forall s \in \mathcal{S}. \quad (5) \]

The problem of an uninformed investor is to choose \( B^U(s) \) for all \( s \in \mathcal{S} \) to maximize (1) subject to (2), (3) and (5) for the UP auction, and (2), (4) and (5) for the DP auction.

The problem of an informed investor, given that he knows the realized quality shock \( \theta \in \{g, b\} \), is higher dimensional as he has to determine bids \( B^I(s, \theta) \) for each state \( s \in \mathcal{S} \) conditional on the realized \( \theta \) so as to maximize

\[ \int_{\eta} \{ U(B_{RF}^I([\theta, \eta], \theta)) \kappa_\theta + U(B_{RF}^I([\theta, \eta], \theta) + B_R^I([\theta, \eta], \theta)) (1 - \kappa_\theta) \} g(\eta) d\eta \quad \forall \theta \in \{g, b\}, \quad (6) \]

where the total purchases of the risky bond for each realized \( \theta \) is

\[ B_R^I(s, \theta) = \sum_{s': P(s') \geq P(s)} B^I(s', \theta) \quad \forall \theta \in \{g, b\}, \]

the total, auction-specific, purchases of the risk-free bond for each realized \( \theta \) is,

**UP auction**

\[ B_{RF}^I(s, \theta) = W - \left[ \sum_{s': P(s') \geq P(s)} B^I(s', \theta) P(s) \right] \quad \forall s \in \mathcal{S} \text{ and } \forall \theta \in \{g, b\}, \]

**DP auction**

\[ B_{RF}^I(s, \theta) = W - \left[ \sum_{s': P(s') \geq P(s)} B^I(s', \theta) P(s') \right] \quad \forall s \in \mathcal{S} \text{ and } \forall \theta \in \{g, b\}, \]

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and the nonegativity constraints are

\[ B^I(s, \theta) \geq 0 \quad \text{and} \quad B^I_{RF}(s, \theta) \geq 0 \quad \forall s \in S \quad \text{and} \quad \forall \theta \in \{g, b\}. \]

Trivially, just as no investor should bid at prices that are not marginal conditional of his information, the informed investor should only bid at prices \( P(\theta, \eta) \) and not at prices \( P(\theta', \eta) (\theta' \neq \theta) \) given that he knows the quality shock is \( \theta \).

**Bid overhang constraint.** Because the marginal price \( P(s) \) is defined to be the highest price such that demand is enough to cover the government’s supply of debt, bids and prices must also satisfy an additional constraint that we call the bid-overhang constraint. This constraint requires that there cannot exist a state \( \tilde{s} \) such that \( P(\tilde{s}) > P(s) \), and at the marginal price \( P(\tilde{s}) \), there is enough demand to cover the supply in state \( s \). As per our auction protocol, even though \( P(s) \) may satisfy market clearing in state \( s \) it cannot be an equilibrium price as the government would be able to raise those funds at a higher price, namely \( P(\tilde{s}) \).

**Definition 2.** Formally, the bid-overhang constraint is the requirement that

\[ \text{for any } s \in S \text{ and for all } \tilde{s} \in S : P(\tilde{s}) > P(s), \]

\[ \text{UP : } (1 - \eta(s)) \left\{ \begin{array}{l} (1 - n) \left[ \sum_{s' : P(s') \geq P(\tilde{s})} B^U(s') \right] \\ + n \left[ \sum_{s' : P(s') \geq P(\tilde{s})} B^I(s', \theta(s)) \right] \end{array} \right\} P(\tilde{s}) < D \]

\[ \text{DP : } (1 - \eta(s)) \left\{ \begin{array}{l} (1 - n) \left[ \sum_{s' : P(s') \geq P(s)} B^U(s') P(s') \right] \\ + n \left[ \sum_{s' : P(s') \geq P(\tilde{s})} B^I(s', \theta(s)) P(s') \right] \end{array} \right\} < D. \]

Notice that \( P(s) \) is obtained from market clearing in state \( s \) when evaluating the demand at \( P(s) \) and the same for \( P(\tilde{s}) \). If \( P(\tilde{s}) > P(s) \), what this constraint implies is that there cannot be excess demand in state \( s \) when the demand is evaluated at \( P(\tilde{s}) \).

One can see from inspection that this constraint cannot bind in the DP auction since the price at which a bid is executed is fixed by the bid price. Therefore, in the DP auction the demand in state \( s \) when evaluated at \( P(\tilde{s}) \) is always smaller than when evaluated at \( P(s) \) when \( P(\tilde{s}) > P(s) \). The bid-overhang constraint, however, can and does bind in the
The reason is that in the UP auction the demand in state \( s \) when evaluated at \( P(\tilde{s}) \) can be larger than when evaluated at \( P(s) \) when \( P(\tilde{s}) > P(s) \), as all bidders pay a larger price, \( P(\tilde{s}) \), for all their bids.

**Definition 3.** An equilibrium of our auction model is defined as a function \( P : S \to [0, 1] \), and bidding functions \( B^U : S \to [0, W] \) and \( B^I : S \times \{ g, b \} \to [0, W] \), such that

1. each type of investor’s bid function solves their problem,
2. the market clearing condition is satisfied for all \( s \in S \), and
3. the bid-overhang constraint is satisfied at each \( s \in S \).

**Remark 3.** Formulating an equilibrium in this stark fashion where bids are defined as functions of the state is isomorphic to a more standard formulation where bids are defined as functions of prices as well. The price functions are the same, as \( B^U(P(s)) = B^U(s) \) and 0 elsewhere, while \( B^I(P(s), \theta) = B^I(s, \theta) \) and 0 elsewhere. The main difference is that the standard formulation defines the bid function over all potential prices rather than marginal prices only. This standard formulation is poorly behaved in the sense that the bid function is discontinuous around marginal prices because no investor bids on non-marginal prices. This is particularly problematic when \( \eta \) is discrete.

**Proposition 1.** For both the UP and the DP auctions the price function \( P(\theta, \eta) \) is decreasing in \( \eta \). Hence a bid made at a price \( P(\theta, \eta) \) is in-the-money for all \( \hat{\eta} \geq \eta \) given \( \theta \), and if there exists an \( \bar{\eta} \), given \( \bar{\theta} \neq \theta \), such that \( P(\bar{\theta}, \bar{\eta}) = P(\theta, \eta) \), then it is in-the-money for all \( \hat{\eta} \geq \bar{\eta} \) given \( \bar{\theta} \). Because the price schedule conditional on \( \theta \) is bounded and monotonic, it follows that it is both continuous and differentiable almost everywhere.

**Proof.** For the DP auction this follows directly from the market clearing condition. For the UP auction it follows from the debt-overhang constraint.

### 3.1 Comparison of Auction Equilibrium and Competitive Equilibrium

Because risky bonds are always sold at a common price in the UP auction it is in many cases isomorphic to a standard competitive equilibrium with heterogeneous information.
We establish this link here. No such equivalence between auction equilibria and competitive equilibria holds for the DP auction model since by construction risky bonds are selling at multiple prices.

In the UP auction, the uninformed investor problem is choosing $B_U(s)$ to maximize 
\[
(1) \quad \text{subject to (2), (3) and the nonegativity constraint (5).}
\]
In the competitive equilibrium his problem is identical with the only difference that the nonnegativity constraint applies to the total purchases of the risky and the risk-free bond at each state $s$, and we replace constraint (5) with
\[
B_R^U(s) \geq 0 \quad \text{and} \quad B_{RF}^U(s) \geq 0 \quad \forall s \in S. \tag{8}
\]
The same considerations hold for the informed investor.

To relate constraints (5) and (8), from a UP auction equilibrium we can construct the associated total risky bond purchases by summing over the in-the-money bids. To go the other way and construct the associated state-by-state bids given the bond purchases in a competitive equilibrium, \{\(B^I_R(s, \theta), B^U_R(s)\}\), we can use the difference between the risky bond purchases at $s$ and those at the next highest price $s'$; this is
\[
B^U(s) = B^U_R(s) - B^U_R(s'), \tag{9}
\]
where $P(s') = \min \{P(s'') > P(s)\}$. The same object for the informed is constructed using their conditional total purchases $B^I_R(s, \theta(s))$. If these state-by-state bids are nonnegative then the nonnegativity constraint on bids for the UP auction is not violated. As the total risky bond purchase will be nonnegative, there is an associated competitive equilibrium in this case. Inversely, if the state-by-state bids are such that the nonnegativity constraint on bids for the UP auction is violated, there is an associated competitive equilibrium only when the total risky bond purchases also bind and are zero. This discussion implies that there is no associated competitive equilibrium when there are states for which the nonnegativity constraint does not bind for total purchases in the competitive equilibrium, but do bind in particular states in the UP auction. We summarize this discussion in the following proposition.

**Proposition 2.** A UP auction equilibrium \(\{P(s), B^I(s, \theta), B^U(s)\}\) in which the nonnegativity
constraint on risky bond purchases only binds when total risky bond purchases are 0 for all \( s \in S \) has an associated competitive equilibrium \( \{ P(s), B_{CE}^U(s), B_{CE}^I(s, \theta) \} \). Any competitive equilibrium \( \{ P(s), B_{CE}^I(s, \theta), B_{CE}^U(S) \} \) in which the associated bids \( \{ B^I(s, \theta), B^U(s) \} \) constructed using (9) are nonnegative and satisfy the bid-overhang constraint (7) is an auction equilibrium.

The bid-overhang constraint can lead to the set of auction equilibria being a subset of the set of competitive equilibria. In the numerical examples, we will illustrate how this can happen.

### 3.2 Investor’s Optimal Bid Policies

We turn next to explicitly characterizing the optimal bid policies for the investors in our two auction protocols. An important input to the investor problem is an inference problem determining the quality of the bond an investor expects receive conditional on a given bid being executed. We start first with the informed investor’s problem since he knows \( \theta \) and hence does not have an inference problem.

#### 3.2.1 Informed Investor

The informed investors problem is the simplest because they know the true quality shock to the bond, which corresponds to one of the components of the state \( s, \theta(s) \). Hence informed investors do not have an inference problem. His f.o.c. for his bids \( B_{RF}^I([\theta^*, \eta^*]) \) for each state \( s = [\theta^*, \eta^*] \) at a UP auction is given by

\[
\int_{\eta} \left\{ \begin{array}{c}
-U'(B_{RF}^I([\theta^*, \eta])) \kappa_{\theta^*} P([\theta^*, \eta]) \\
+U'(B_{RF}^I([\theta^*, \eta])) (1 - \kappa_{\theta^*})(1 - P([\theta^*, \eta]))
\end{array} \right\} \mathcal{I} \{ P([\theta^*, \eta]) \geq P([\theta^*, \eta]) \} g(\eta) d\eta
\]

\[ -\chi^I([\theta^*, \eta^*]) = 0, \]

where \( \chi^I(s) \) is the multiplier on the nonnegativity constraint, and \( \mathcal{I} \{ \cdot \} \) is an indicator function. In this condition, the price \( P([\theta^*, \eta]) \) at which the investor buys the risky bond plays two roles. On the one hand, it determines the set of states in which his purchase is
in-the-money through the indicator function. On the other hand, it also determines the price at which the investor purchases this bond since the investor pays the marginal price $P([\theta^*, \eta])$ on all possible demand shocks $\eta$ in which his bid is in the money.

For the DP auction the f.o.c. can be expressed as

$$
\int_{\eta} \left\{ -U'(B_{RF}'([\theta^*, \eta])) \kappa_{\theta^*} P([\theta^*, \eta]) \\
+ U'(B_{RF}'([\theta^*, \eta])) \left( 1 - \kappa_{\theta^*} \right) \left( 1 - P([\theta^*, \eta^*]) \right) \\
+ B_R([\theta^*, \eta]) \right\} \mathbf{1}\{P([\theta^*, \eta^*]) \geq P([\theta^*, \eta])\} g(\eta) d\eta
$$

$$
-\chi^I([\theta^*, \eta^*]) = 0,
$$

In contrast to the UP auction, in the DP auction the price $P([\theta^*, \eta])$ only determines the set of states the investor is in the money, as the investor always pays the price of the bid, $P([\theta^*, \eta^*])$ when it is in the money.

### 3.2.2 Uninformed Investor

Next we turn to an uninformed investors inference problem. First, we rewrite the problem in terms of the believed probability of default conditional on observing a marginal price.

**Inference:** Given that bids are executed depending on the realization of the marginal price and that the quality of a bond is fully pinned down by its default probability, this inference problem is equivalent to computing the expected default probability of a bond given the realization of a marginal price. We denote this conditional expected default probability by $\tilde{\kappa}$. For the informed, $\tilde{\kappa}(P(\theta, \eta) | \theta) = \kappa_{\theta}$ because they know the true $\theta$. For the uninformed there are two cases:

1. For any $(\theta, \eta)$ such that $\not\exists (\theta', \eta')$ with $P(\theta, \eta) = P(\theta', \eta')$, then $\tilde{\kappa}(P(\theta, \eta)) = \kappa_{\theta}$.

2. If there are two states $(\theta, \eta)$ and $(\theta', \eta')$ such that $P(\theta', \eta') = P(\theta, \eta)$ and $\theta' \neq \theta$ the solution to the uninformed investor’s inference problem is as follows.

Given $P(\theta, \eta)$, define $\eta = \phi(P | \theta)$, where $\phi$ is the inverse function of the price with
respect to $\eta$\textsuperscript{10}. Define the probability of an interval of prices $P \subset \mathcal{P}$ conditional on $\theta$ as
\[
h(P|\theta) = \int_{\{\eta: P(\theta, \eta) \in P\}} g(\eta) d\eta = \int_{\tilde{P} \in P} g(\phi(\tilde{P}|\theta)) \frac{\partial \phi(\tilde{P}|\theta)}{\partial \tilde{P}} d\tilde{P}.
\]
Note that the slope of the inverse function with respect to the price determines the size of the set of $\eta$’s that are associated with the prices in $P$ (given $\theta$). The unconditional probability of the set of prices is then given by
\[
h(P) = \sum_{\theta} f(\theta) h(P|\theta),
\]
and the probability of $\theta$ conditional on a price in $P$ is simply $f(\theta) h(P|\theta)/h(P)$. We can define the probability density function of a particular price $P \in \mathcal{P}$, given $\theta$, by shrinking the set $P \to P$, and observing $h$ in the limit, or
\[
Pr(P|\theta) = \lim_{P \to P} \frac{h(P|\theta)}{\Delta(P)},
\]
where $\Delta(P)$ is the length of the price interval. This then leads to the inferred default probability
\[
\tilde{\kappa}(P) = \frac{\sum_{\theta} f(\theta) Pr(P|\theta) \kappa_{\theta}}{\sum_{\theta} f(\theta) Pr(P|\theta)}.
\]

**Uninformed Investor’s Problem:** Given this inference problem, and regrouping states according to marginal prices, the uninformed investor’s payoff can be expressed in terms of the expected probability of default as
\[
\sum_{\theta \in \{g, b\}} \int_{\eta} \left\{ \begin{array}{c} U(B^{U}_{RF}([\theta, \eta])) \tilde{\kappa}(P([\theta, \eta])) \\ + U(B^{L}_{RF}([\theta, \eta]) + B^{U}_{RF}([\theta, \eta])) (1 - \tilde{\kappa}(P([\theta, \eta]))) \end{array} \right\} f(\theta) g(\eta) d\eta.
\]
We start with characterizing the optimal bid policy of an uninformed investor at a UP

\textsuperscript{10}Note from proposition 1 that $P(\theta, \eta)$ is continuous almost everywhere and since rationing does not occur in equilibrium, strictly monotonic, thus it is invertible almost everywhere. See Rudin (1964, p. 90).
auction. The uninformed investor’s f.o.c. for \( B^U([\theta^*, \eta^*]) \) at a UP auction is given by

\[
\sum_{\theta \in \{g,b\}} \int_{\eta} \left\{ -U'(B^U_{R,F}([\theta, \eta]))\tilde{\kappa}(P([\theta, \eta])P([\theta, \eta])) \\
+ U'(B^U_{R,F}([\theta, \eta])) \left(1 - \tilde{\kappa}(P([\theta, \eta]))(1 - P([\theta, \eta]))\right) \right\} \times \\
I \{ P([\theta^*, \eta^*]) \geq P([\theta, \eta])\} f(\theta) g(\eta) d\eta - \chi^U([\theta^*, \eta^*]) = 0.
\]

The uninformed investor’s f.o.c. for \( B^U(s) \) at a DP auction is given by

\[
\sum_{\theta \in \{g,b\}} \int_{\eta} \left\{ -U'(B^U_{R,F}([\theta, \eta]))\tilde{\kappa}(P([\theta, \eta])P([\theta^*, \eta^*])) \\
+ U'(B^U_{R,F}([\theta, \eta])) \left(1 - \tilde{\kappa}(P([\theta, \eta]))(1 - P([\theta^*, \eta^*]))\right) \right\} \times \\
I \{ P([\theta^*, \eta^*]) \geq P([\theta, \eta])\} f(\theta) g(\eta) d\eta - \chi^U([\theta^*, \eta^*]) = 0.
\]

4 Characterization of UP Auction Equilibrium

We start with the uniform-price auction. Recall that the fraction of informed investors was denoted by \( n \), and that this fraction determines the degree of asymmetric information in our model. Accordingly, we will use \( P^{UP}(s; n) \) to denote the price function for each state and level of asymmetric information in the UP auction. When there is no risk of confusion, we sometimes also simply write \( P(s) \). In a similar fashion, we will use \( B^{UP,U}(s; n) \) for the bids of the uninformed and \( B^{UP,I}(s, \theta; n) \) for the bids of the informed, but may sometimes use the simpler notation \( B^U(s) \) and \( B^I(s, \theta) \) when there is no risk of confusion.

4.1 Symmetric Information Benchmarks

We begin by considering the two symmetric information benchmarks: the symmetric ignorance equilibrium in which no investor is informed \( (n = 0) \), and the symmetric information equilibrium in which all investors are informed \( (n = 1) \).
4.1.1 Symmetric Ignorance

Since there are no informed investors in the symmetric ignorance equilibrium, marginal prices cannot depend upon $\theta$, and we must have $P(g, \eta) = P(b, \eta)$ for all $\eta \in \mathcal{H}$. Hence, we can simplify our notation and write $P(\eta)$ for prices and $B(\eta)$ for bond purchases. Because $P(\eta)$ is declining in $\eta$, the in-the-money set for bids $B(\eta)$ is $[\eta, \eta_M]$, this is, the investor gets to buy $B(\eta)$ whenever the demand shock is in the set $[\eta, \eta_M]$. It follows also that when the demand shock is $\eta$, all bids $B(\hat{\eta})$ for $\hat{\eta} \leq \eta$ are in-the-money.

As market-clearing can be stated as

$$D = (1 - \eta) \left[ \int_{0}^{\eta} B(\hat{\eta}) d\hat{\eta} \right] P(\eta).$$

(10)

It follows from the market-clearing condition that $B(\eta) > 0$, and hence the short-sale constraint cannot bind for any $\eta$. Because prices cannot convey any information about $\theta$, it follows that $\tilde{\kappa}(P) = \kappa^U = f(g)\kappa_g + f(b)\kappa_b$, which is the ex-ante probability of default for all $P$. The f.o.c. for our uninformed investor (all investors in this case) bidding in state $\eta^*$, and since the bid $B(\eta^*)$ is in the money when $P(\eta) < P(\eta^*)$ and then $\eta > \eta^*$, is

$$\int_{\eta^*}^{\eta_M} \left\{ \begin{array}{c} -U'(B_{RF}(\eta))\kappa^U P(\eta) \\ +U'(B_{RF}(\eta) + \left[ \int_{0}^{\eta} B(\hat{\eta}) d\hat{\eta} \right] (1 - \kappa^U)(1 - P(\eta)) \end{array} \right\} g(\eta) d\eta = 0.$$

From the market-clearing condition, it follows that

$$\left[ \int_{0}^{\eta} B(\hat{\eta}) d\hat{\eta} \right] P(\eta) = \frac{D}{1 - \eta}$$

and from the budget constraint the purchase of risk-free bonds is

$$B_{RF}(\eta) = W - \frac{D}{1 - \eta}.$$

Hence, the f.o.c. reduces to a condition on prices $P(\eta)$. This result combined with the fact that the term in brackets is 0 for any interval $[\eta^*, \eta_M]$ implies that it is also 0 at each $\eta$. Thus we can solve for the equilibrium price separately at each $\eta$. 

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Example 1. If we assume log preferences, then our equilibrium can be solved in closed form. The first-order condition becomes
\[
\frac{(1 - \kappa^U)[1 - P(\eta)]}{W - \frac{D}{1-\eta} + \frac{D}{1-\eta} P(\eta)} - \frac{\kappa^U P(\eta)}{W - \frac{D}{1-\eta}} = 0.
\]
which implies
\[
P(\eta) = \frac{(1 - \kappa^U)W - \frac{D}{1-\eta}}{W - \frac{D}{1-\eta}} = 1 - \kappa^U - \frac{\kappa^U D}{1-\eta},
\]
and that
\[
\left[ \int_{0}^{\eta} B(\hat{\eta})d\hat{\eta} \right] = \frac{W \left[ 1 - \kappa^U - P(\eta) \right]}{P(\eta)[1-P(\eta)]}.
\]
By inspection we can see that \( P(\eta) \) is equal to the risk-free price \( 1 - \kappa^U \) if \( D = 0 \), continuous and declining in \( D/(1-\eta) \).

4.1.2 Symmetric Information

We now consider the symmetric information equilibrium in which all investors are informed \((n = 1)\). Since all investors bid contingent on \( \theta \), the equilibrium can be determined conditional on the realized quality shock \( \theta \). The logic is just as in the symmetric ignorance case, only now the default probability is \( \kappa_{\theta(s)} \). Thus the f.o.c. for an informed investor is bidding in state \((\theta(s), \eta^*)\) is similar to that above, and we can again appeal to the market-clearing condition and the budget constraint to get a simple condition which we can solve for the price \( P([\theta(s), \eta^*]) \)
\[
\int_{\eta^*}^{\eta_M} \left\{ -U' \left( W - \frac{D}{1-\eta} \right) \kappa_{\theta(s)} P([\theta(s), \eta]) \right. \\
+ U' \left( W - \frac{D}{1-\eta} + \frac{D}{1-\eta} P([\theta(s), \eta^*]) \right) (1 - \kappa_{\theta(s)})(1 - P([\theta(s), \eta])) \left\} g(\eta)d\eta = 0.
\]
Example 2. If we again assume log preferences the closed-form conditional price schedules is
\[
P(\theta, \eta) = 1 - \kappa_\theta - \frac{\kappa_{\theta} D}{W - \frac{D}{1-\eta}}.
\]
Observe that the \( \theta = g \) schedule will lie above the \( \theta = b \) conditional on \( \eta \), and that both schedules
are continuous and declining in $D/(1-\eta)$ just as in the $n=0$ case. Going forward, an important question will be whether the $\theta = g$ schedule also lies above the $\theta = b$ schedule for all $\eta$. It is easy to verify that this is the case if

$$1 - \kappa_g - \frac{\kappa_g D}{W - \frac{D}{1-\eta_M}} > 1 - \kappa_b - \frac{\kappa_b D}{W - D}. \quad (11)$$

If this condition is satisfied, then each marginal price is associated with a unique $\theta$, and there is perfect ex-post inference. If not, then the price schedules overlap and some marginal prices will be associated with both values of $\theta$. From inspection one can see that overlap is more likely the closer are the bankruptcy costs and the bigger is the span of the demand shocks.

### 4.2 Replication

What are the gains to be informed? If prices are fully revealing, then in a competitive equilibrium, there are no gains. A similar condition holds in the UP auction under somewhat more stringent conditions.

**Proposition 3.** In a UP auction the uninformed will be able to replicate the total bids of the informed state-by-state, and hence their ex-ante payoff, if:

1. Each marginal price is associated with a unique state in $S$.

2. When we order the marginal prices and the associated total bids of the informed from the highest to the lowest marginal price, the total bids of the informed are also weakly ranked from lowest to highest.

**Proof.** The first condition ensures that the uninformed can compute the state associated with each marginal price, and even though not observing the marginal price ex-ante, being able to bid at those prices accordingly. The second condition ensures that the no short-sale constraint does not bind, so the uninformed investor can optimize in each state with respect to total bond purchases, knowing the default probability at which these will be executed. When this is true, the uninformed can replicate the payoff of the informed by matching their total bond purchases to those of the informed in each state $s$. □
Corollary 4. For any equilibrium with \( n > 0 \), \( P(g, \eta_M, n) > P(b, 0, n) \) and \( B^I_R([g, \eta_M], g; n) < B^I_R([b, 0], b; n) \) are sufficient conditions for the uninformed to be able to completely replicate the portfolio of the informed state-by-state, and hence their payoffs.

Note that the uninformed will not make the same marginal bids as the informed in each state, but they will buy the same total amount as the informed in each state, achieving perfect portfolio replication. The reason is all of uninformed’s bids on the \( \theta = g \) schedule are accepted when \( \theta = b \). This is not the case for the informed, whose bids are conditioned on \( \theta \). Thus, the uninformed make the same marginal bids on the high-price schedule, but make lower marginal bids on the low-price schedule in order to achieve the same total bids both in the high and low price schedules.

Example 3. Returning to our log example, assume that condition (11) is satisfied so that the price schedules do not overlap in the symmetric information case. To verify that we can have replication (a single uninformed investor can replicate the portfolio and utility of informed investors and therefore there is no cost of being uninformed) when \( n = 1 \) we only need to check that \( B^I_R([g, \eta_M], g; 1) < B^I_R([b, 0], b; 1) \). The equilibrium total bond purchases of the informed in the fully informed equilibrium are

\[
B^I([\theta, \eta], \theta, 1) = \frac{W \kappa_\theta \frac{D}{W - \frac{D}{1-\eta}}}{\left(1 - \kappa_\theta - \frac{\kappa_\theta D}{W - \frac{D}{1-\eta}}\right)\left(\kappa_\theta + \frac{\kappa_\theta D}{W - \frac{D}{1-\eta}}\right)} = \frac{\frac{D}{1-\eta}}{(1 - \kappa_\theta)W - \frac{D}{1-\eta}}.
\]

We can use this result to evaluate whether the second condition in proposition 3 holds and then whether perfect replication is possible in the informed equilibrium for different parameter values. Indeed it is easy to generate such outcome when the default probabilities are sufficiently different and the span of the demand shocks is not too large.

What happens as we reduce \( n \) below 1 in a case where there is perfect replication? At first nothing because the total bids of the uninformed are the same as informed. In fact this continues up until the bid overhang constraint binds between \( P([g, \eta_M]) \) and \( P([b, 0]) \). Since per capita expenditures must equal \( D/(1 - \eta) \), and since the uninformed are replicating the total bids of the
informed, it follows that the bid overhang constraint will bind when

\[(1 - n) \frac{D}{1 - \eta_M} \geq D, \text{ or } n \leq \eta_M.\]

In words, for all \(n > \eta_M\) we need the bids from the informed agents for the government to raise \(D\) in the state \((b, 0)\) (when the bond is bad but all investors show up in the auction), as the fraction of uninformed investors is low. Once there are enough uninformed investors, the government does not need the bids of informed investors in such state, and the bid-overhang constraint binds.

4.3 Asymmetric Information

The fact that we can characterize our log example in the case where all of the prices is distinct echoes the result in the competitive equilibrium literature that generically competitive equilibria will exist when there is perfect revelation of information. We now turn to the surprising result that we can do this even when the debt-overhang constraint binds and therefore cannot be perfect revelation ex post.

**Example 4.** What happens in our log example when there is perfect replication by the uninformed in the symmetric information case (this is, condition (11) is satisfied so that the fully informed price schedules do not overlap and \(B^l_R([g, \eta_M] ; g; 1) < B^l_R([b, 0] , b; 1)\)), but the number of informed falls so low that \(n \leq \eta_M\) and the bid overhang constraint starts binding?

Once the bid-overhang constraint binds, some prices must overlap. Consider two levels of the demand shock, \(\eta_g\) and \(\eta_b\), which must share a common price \(P\), then we have three conditions that must hold:

1. We need to clear the market in the \(\theta = g\) state

\[n \left( \frac{1 - \kappa_g - P}{1 - P} \right) + (1 - n) \left( \frac{1 - \tilde{\kappa} - P}{1 - P} \right) = \frac{D}{W} \frac{1}{1 - \eta_g},\]

2. We need to clear the market in the \(\theta = b\) state

\[n \max \left[ \left( \frac{1 - \kappa_b - P}{1 - P} \right), 0 \right] + (1 - n) \left( \frac{1 - \tilde{\kappa} - P}{1 - P} \right) = \frac{D}{W} \frac{1}{1 - \eta_b}.\]
Note that the short sale constraint will bind on the informed when

\[ P \geq 1 - \kappa_b. \]

3. Bayesian updating implies that \( \tilde{\kappa} \) determines the ratio of demand shocks in the good and the bad states or

\[ \tilde{\kappa} = \frac{(\#g)f(g)\kappa_g + (\#b)f(b)\kappa_b}{(\#g)f(g) + (\#b)f(b)}, \]

or

\[ r(P) = \frac{(\#b)}{(\#g)} \cdot \frac{f(b)[\tilde{\kappa} - \kappa_g]}{f(g)[\kappa_b - \tilde{\kappa}]} .\]

For this application of Bayesian updating we assume that the demand shock, \( \eta \), is in a grid uniformly distributed and with a grid size of \( \varepsilon \).

Imagine \( n = \eta_g \). As we have discussed the debt overhand constraint binds between \([g, \eta_g]\) and \([b, 0]\). Starting from these two points and denoting \([b, 0]\) generically as \([b, \eta_b]\), we can solve conditions (1 – 3) to determine \( P = P(g, \eta_g) = P(b, \eta_b) = \bar{\kappa}(P) \) and \( r(P) \). Then, we increase \( \eta_g \) to \( \eta_g + \varepsilon \) and \( \eta_b \) to \( \eta_b + r(P)\varepsilon \). (Note that we are endogenously determining the grid position on the \( \theta = b \) schedule which we can denote by \( \eta_b(\eta_g) \)). This initial problem has a very simple solution when the short-sale constraint binds since this implies that \( P = P(g, \eta_g; n = 1) \), \( \bar{\kappa} = \kappa_g \) and \( r(P) = 0 \). If this is not the case then \( P \) will be lower and hence \( \bar{\kappa} < \kappa_b \). Moreover, as the gap between \( \eta_b \) and \( \eta_g \) get larger, \( P \) and \( \bar{\kappa} \) both fall.

We can maintain this construction until \( \eta_b \) on the \( \eta \) grid hits \( \eta_M \), and the overlap in the price schedule ends. The upper schedule is unchanged up until \( \eta_g = n \), while that on the lower schedule is unchanged for all \( \eta_b > \eta_h(\eta_M) \). So long as the short-sale constraint does not bind anywhere, this completely determines the equilibrium of our UP auction model for this value of \( n \).

We already have shown that when \( n > \eta_M \) the price schedule and total bid schedules coincide with the informed equilibrium. Now we consider the limit to the other extreme based on the previous construction. As \( n \to 0 \), the first point on the \( \theta = g \) schedule at which overlap occurs converges to 0, and hence it converges to the first point on the \( \theta = b \) schedule. When \( \eta_g = \eta_b \) and \( n \approx 0 \), then one can see from inspection that the two equations for \( P \) and \( \bar{\kappa} \) converge to those of the uninformed equilibrium. Hence, the uninformed equilibrium price schedule \( P^U(\eta) \) and ex
ante default probability $\kappa_U$, are the solution in the limit. This also implies that $r(P) = 1$ and this convergence happens almost everywhere along the schedule (with the two possible outliers being at the supports of $\eta$).

Now that we have shown how to characterize and in fact compute the equilibria of our model for any $n$ given that we started with non-overlapping price schedules and perfect replication at $n = 1$, we can extend this analysis to the cases in which these conditions are violated.

**Example 5.** Consider the case in which the price schedules overlap for the symmetrically informed case. Then the first price at which they overlap determines the initial values of $\eta_g$ and $\eta_b$. The construction then follows as before assuming the short-sale constraint does not bind on the uninformed.

**Example 6.** Here we discuss what happens if the short-sale constraint binds for the uninformed investor and how to determine the range over which it binds. Once this range has been determined we simply impose 0 new bids for those states, and then solve for the bids at the other prices just as before. This is feasible considering that the range is an interval centered around the point where the two schedules first begin overlapping as the accumulated purchases for the uninformed on the good schedule generate too large purchases at the top of the bad schedule when we ignore the short sale constraint.

For the case in which there is no overlap in the price schedule, this is particularly simple. Binding will occur over intervals of the form $[\eta_g, \eta_M]$ on the $\theta = g$ schedule and $[0, \eta_b]$ on the $\theta = b$ schedule. In this case, the total risky bond purchases will be equal at the endpoints at which we solve using the f.o.c. ignoring the short-sale constraint; i.e. $B^U(g, \theta_g) = B^U(b, \theta_b)$. In addition, the integral of the f.o.c. over these ranges will equal 0, as $s = (g, \eta_g)$ is the point at which the short-sale constraint starts binding. It is easy to see that starting from any $s = (g, \eta_g + \epsilon)$ the integral will turn negative, and extending the integral beyond $(b, \eta_b)$ will turn it positive.

Note that when this occurs, an auction equilibrium of the UIP model no longer has an associated competitive equilibrium, as this is a case of nonegativity constraints binding for bids in a particular range but not binding for total purchases on that range.
This completes our characterization of the log example. We used log because it allowed for some close form results. However, nothing in the characterization hinged on anything beyond concavity. We illustrate this construction and its results using a numerical version of this example later.

4.4 Convergence to Symmetric Information and Ignorance

This section generalizes the convergence result to the symmetric ignorance case as $n \to 0$ that we discussed briefly in the previous construction of the log example for UP auctions to any utility function and for both auction protocols.

The following proposition is a characterization of the UP auction equilibrium in the symmetric information case and follows from the same logic as the example.

**Proposition 5.** If in the informed equilibrium the price function satisfies $P([g, \eta_M]; n = 1) > P([b, 0]; n = 1)$ and the total bids of the informed $B^I([g, \eta_M], g, 1) < B^I([b, 0], b, 1)$ then the informed equilibrium price function and total bids are an equilibrium outcome for any $n > \eta_M$ and there is complete replication by the uninformed.

We turn now to the other extreme. When $n \to 0$, then almost all expenditures at the auction, which we denote by $X^U(\theta, \eta; n)$, must be made by the uninformed type. This implies that

$$X^U(\theta, \eta; n) \to D/(1 - \eta) \text{ as } n \downarrow 0,$$

under both auction protocols. When $n$ is sufficiently close to 0, it must therefore be the case that

$$X^U(\theta, \eta; n) > X^U(\theta', \eta' : n) \quad \text{if } \eta > \eta'.$$

But this in turn implies that

$$P(\theta, \eta) < P(\theta', \eta').$$

This is because the cumulated expenditures of the uninformed more than cover demand at $(\theta', \eta')$ at price $P(\theta, \eta)$ hence the price $P(\theta', \eta')$ cannot be lower $P(\theta, \eta)$. Note that this is true under either auction protocol. From this we get the following result.
Proposition 6. The price schedules $P([g, \eta]; n)$ and $P([b, \eta]; n)$ converge to each other (for interior $\eta$) since for $n$ sufficiently close to 0, $\eta$ must partially order the price schedules $P(\theta, \eta)$ under both of our auction protocols: i.e. if $\eta > \eta'$ then $P([\theta, \eta]; n) < P([\theta', \eta']; n)$.

This proposition implies that prices must be sorted by $\eta$ when $n$ is small. Thus, prices must lie between the low price at the small $\eta$ and the high price at the higher $\eta$. When $\eta$ is a continuous interval, this implies that the price schedules must converge at every interior point in which the price schedules are continuous. Even when $\eta$ is not an interval, it follows that the uninformed investor faces the same set of in-the-money states when he buys along the high price schedule $P(g, \eta)$ as he does when buying at $P(\eta)$ in the uninformed equilibrium. In the DP auction protocol, the price that will cause the uninformed to spend $D/(1 - \eta)$ converges to $P(\eta)$ everywhere but at the bottom when $\eta = \eta_M$.

5 Characterization of DP Auction Equilibrium

5.1 Symmetric Information Benchmarks

We start analyzing the DP auction with the two symmetry benchmarks, symmetric ignorance ($n = 0$) and symmetric information ($n = 1$). With symmetric ignorance we can again simplify our notation to have $P(\eta), B(\eta), X(\eta)$ and $R(\eta),$. With this change our market clearing condition (10) becomes

$$D = \int_0^\eta B(\hat{\eta})P(\hat{\eta})d\hat{\eta}.$$ 

Note the first important difference between UP and DP auctions: expenditures and hence the clearing condition in the DP auction do not just depend upon the total number of bids, but also on the *prices* at which the individual bids are executed. Because of this, total expenditures *must* be monotonically increasing in $\eta$. Hence the bid-overhang constraint cannot bind. Market clearing implies that the bids must always be positive, i.e. $B(\eta) > 0$ for all $\eta \in \mathcal{H}$. This in turn implies that the short-sale constraint cannot bind for any $\eta$. The
first-order condition for the uninformed investor at $\eta^*$ can be expressed as

$$
\int_{\eta^*}^{\eta_M} \left\{ -U' \left( W - \int_0^\eta B(\hat{\eta}) P(\hat{\eta}) d\hat{\eta} \right) \kappa^U P(\eta^*) \\
+ U' \left( W + \int_0^\eta B(\hat{\eta}) [1 - P(\hat{\eta})] d\hat{\eta} \right) (1 - \kappa^U) [1 - P(\eta^*)] \right\} g(\eta) d\eta = 0. \tag{12}
$$

Notice a second key difference relative to the UP auction: the cumulation of the marginal utilities are being multiplied by the bid price $P(\hat{\eta})$ and the bid return $(1 - P(\hat{\eta}))$ rather than by the marginal prices. This means that the system is not block-recursive as in the UP auction. Instead, it must be solved simultaneously. Moreover, because we are almost everywhere considering the impact of changing a bid on an in-the-money set which contains an interval of $\eta$ values, the set is of positive measure. Hence, inference at a particular price does not have the impact it did for the UP auction.

To solve this problem, we can use the following linear algebra structure, as the short-sale constraints do not bind. Assume that we have a fine grid on the space of $\eta$’s $\{\eta_0, \ldots, \eta_J\}$ which is indexed by $j$ and where $\eta_0 = 0, \eta_J = \eta_M$ and the $\eta$’s are increasing in $j$. Next, we denote the following set of vectors:

$$
\bar{P} = \begin{bmatrix} P(\eta_0) \\ \vdots \\ P(\eta_J) \end{bmatrix}, \quad \bar{B}^U = \begin{bmatrix} B^U(\eta_0) \\ \vdots \\ B^U(\eta_J) \end{bmatrix}, \quad \left(1 - \bar{P}\right) = \begin{bmatrix} 1 - P(\eta_0) \\ \vdots \\ 1 - P(\eta_J) \end{bmatrix}.
$$

And we denote the following set of triangular matrices

$$
P = \begin{cases} 
P_{ij} = P(\eta_i) \text{ if } i \leq j \\
0 \text{ o.w.}
\end{cases}, \quad 1 - P = \begin{cases} 
1 - P_{ij} = 1 - P(\eta_i) \text{ if } i \leq j \\
0 \text{ o.w.}
\end{cases}
$$

Then with this notation, the system of equations can be expressed in vector form as

$$
-U' \left( W - P \times \bar{B}^U \right) \cdot \bar{P} \cdot \kappa^U + U' \left( W + [1 - P] \times \bar{B}^U \right) \cdot \left[1 - \bar{P}\right] \ast [1 - \kappa^U] = 0.
$$

With symmetric information ($n = 1$), we can solve for the equilibrium separately for each $\theta$. This requires replacing $\kappa^U$ with the appropriate conditional default probability $\kappa_\theta$,
then proceeding in the analogous manner thereafter.

An important implication is that an uninformed investor would never be able to replicate the portfolio of informed investors, no matter whether the short-sale constraints bind or not. In other words, under DP auction there is always a cost of being uninformed.

To see why, consider the case where \( P(g, 0) > P(b, 0) \) and an informed investor buys a positive quantity of bonds both at \( P(g, 0) \) when \( \theta = g \) and at \( P(b, 0) \) when \( \theta = b \). If the uninformed investor bids a positive amount at \( P(g, 0) \), then when the state is \( (b, 0) \) he is going to spend \( P(g, 0)B^U(g, 0) + P(b, 0)B^U(b, 0) \), while the informed investor will have spent \( P(b, 0)B^I(b, 0) \). Thus even if \( B^U(g, 0) + B^U(b, 0) = B^I(b, 0) \), and they are both buying the same quantity of bonds, the uninformed is paying more. Then, note that even if \( P(g, 0) = P(b, 0) \) the informed investor will want to alter the quantity he bids in response to the quality shock, which the uninformed cannot do as the price is the same. This leads to the following proposition.

**Proposition 7.** In a DP auction, the uninformed will never be able to replicate the payoffs and bids of the informed so long as:

1. \( \kappa_g \neq \kappa_b \) and \( f(g) \) and \( f(b) \) are both positive

2. the informed investor bids positive amounts for both \( \theta = g \) and \( \theta = b \) for some values of \( \eta \).

### 5.2 Asymmetric Information

When \( n \) is interior and there are positive measures of informed and uninformed, prices \( P([\theta, \eta]; n) \) will convey information about \( \theta \). This is somewhat irrelevant to an informed agent who knows \( \theta \) and has to bid before learning \( \eta \). Hence we can model his behavior just as we did in the case of \( n = 1 \) and solve for his optimal policy accordingly. With a grid on \( \eta \) there will generically be perfect ex post revelation, but this does not fundamentally change the problem of an uninformed agent since they care about the entire set of in-the-money prices when they choose their bids at state \( s \).

Given a grid of \( \eta \in \mathcal{H} \), and prices \( P(s; n) \) we can order the states by \( s_i = [\theta_i, \eta_i] \) by the prices in descending order so that \( P(s_i) > P(s_{i+1}) \). Given this, we can construct the
following set of vectors

\[
\tilde{P} = \begin{bmatrix}
P(s_0) \\
\vdots \\
P(s_J)
\end{bmatrix}, \quad \tilde{B}^U = \begin{bmatrix}
B^U(s_0) \\
\vdots \\
B^U(s_J)
\end{bmatrix}, \quad (1 - \tilde{P}) = \begin{bmatrix}
1 - P(s_0) \\
\vdots \\
1 - P(s_J)
\end{bmatrix}, \quad \kappa = \begin{bmatrix}
\kappa(\theta_0) \\
\vdots \\
\kappa(\theta_J)
\end{bmatrix}
\]

and the following set of triangular matrices

\[
P = \begin{cases}
P_{ij} = P(i) \text{ if } i \leq j \\
0 \text{ o.w.}
\end{cases}, \quad 1 - P = \begin{cases}
1 - P_{ij} = 1 - P(i) \text{ if } i \leq j \\
0 \text{ o.w.}
\end{cases}
\]

With this construction, the system of equations to solve for the bond policy of the uninformed can be expressed in vector form as

\[
-U'(W - P \times \tilde{B}^U) \cdot \tilde{P} \cdot \kappa + U'(W + [1 - P] \times \tilde{B}^U) \cdot [1 - \tilde{P}] \cdot [1 - \kappa] \leq 0
\]

and

\[
B^U(s_i) = 0 \text{ when this inequality is slack.}
\]

If we let the grid on \( \mathcal{H} \) become arbitrarily fine, then our solution will converge arbitrarily close to the true solution (of course the linear algebra conditions will become infinite dimensional).\footnote{When implementing the computation of the equilibria in the numerical illustration later for multiple values of the number of informed, \( n \) using homotopy, we were concerned with the implication of prices on the \( g- \) and \( b- \) schedules changing order. For this reason we smeared a share of the bids at \( [\theta, \eta] \) to \( [\theta', \eta'] \) given by \( F(P(s), P(s'), s, s') = \max \left[ 1 - \frac{P(s') - P([\theta, \eta])}{\epsilon}, 0 \right] \) if \( P([\theta, \eta]) < P(\theta', \eta'), \theta \neq \theta' \) and equal to zero otherwise. We then set \( P_{ij} = F(P(s_i), P(s_j), s_i, s_j)P(s_j) \) in \( P \) and correspondingly in \( 1 - P \). In the computation \( \epsilon \) was small enough that at the computed values these fractions played a minimal role.}

An important difference between the auction protocols is the nature of the inference problem solved by an uninformed investor. This can be illustrated as follows by considering a risk-neutral investor.

Remark 4. Consider the special case of a risk-neutral investor and assume that the short-sale constraint does not bind. The UP investor is concerned with the default probability at the marginal price of his bid \((\bar{B}, \bar{P})\); \(\tilde{\kappa}(\tilde{P})\). The DP investor is concerned with the default probability of the entire
set of states at which his bid \((\bar{B}, \bar{P})\) is in the money; i.e. \(E\left\{\bar{\kappa}(P)|P \leq \bar{P}\right\}.\)

Another important distinction between the DP and UP auctions concerns the bid-overhang constraint, which is critical for preventing perfect replication in UP auctions. We have already seen that the uninformed can never replicate the informed portfolio in a DP auction. In line with this result, the bid-overhang constraint never binds in a DP auction. The reason is that total expenditures are cumulated at the bid price, but not at the marginal price. Total expenditures must therefore be strictly increasing in \(\eta\) no matter the slope of the price schedules, so long as marginal bids are positive. Moreover, marginal bids must be positive to clear the market.

6 Numerical Example

We now use a numerical example to better illustrate the implications of our two different auction protocols. In order to obtain sharp results, we focus on a set of parameters for which: i) there is perfect replication in the UP auction when \(n\) is sufficiently close to 1 and ii) total bond purchases are monotonically decreasing in the price \(P(s)\) for the uninformed, and monotonically decreasing in the price \(P((\theta, \eta))\) for the informed. This implies that short sale constraints only bind in circumstances in which total purchases of the risky bond are 0, hence the UP auction equilibrium satisfies the conditions laid out in proposition 2 and has an associated competitive equilibrium.

These parameters are as follows.

1. Preferences are log (consistent with our theoretical examples).
2. The default probabilities are determined by \(\kappa_g = 0.15, \kappa_b = 0.35\) and \(f(g) = 0.4\).
3. The demand shock \(\eta\) is uniformly distributed on \([0, 0.3]\)
4. The per-capita wealth of lenders is \(W = 250\), and the debt to be rolled over is \(D = 60\).
6.1 Symmetric Benchmarks in UP and DP Auctions

These symmetric cases are interesting under several different interpretations of the model. The first is the one we have proposed above, that they represent cases in which no one and everyone chose to become informed about the quality shock $\theta$. Another interpretation is that $n = 0$ corresponds to the case in which there is no information to be learned about the government’s future default behavior and the default probability is deterministic and equal to $\kappa_U$, while $n = 1$ can be interpreted as the case in which all the information about their behavior is public, coming through the aggregate public shock $\nu$ shock that the previous analysis was implicitly conditioning on.

6.1.1 Symmetric Ignorance

We start by computing the uninformed equilibrium for both the UP and the DP auctions. Both the price schedules and the bid schedules are shown in the first panel of figure 2. For both auction protocols, the price declines modestly with the demand shock, because a reduction in the mass of investors forces each investor to hold a higher per-capita share of exposure to the government’s debt. This increases the required risk premium. Reflecting this increased exposure, the total quantity of bonds purchased by investors increases in $\eta$. More surprising, perhaps, is that equilibrium prices and bids are very similar across the two auction protocols for our set of parameters.

There are still some slight differences. For low demand shocks, marginal prices are higher in the UP auction. This reflects the fact that investors do not have to worry about overpaying relative to the marginal price. For high values of the demand shock, marginal prices are higher in the DP auction. This reflects the fact that bids submitted at lower demand shocks are executed at higher prices than in the UP auction, so that a larger share of the debt $D$ has already been sold. Consistent with this logic, the total bid schedules cross in a similar manner.
6.1.2 Symmetric Information

Given that prices are so similar under symmetric ignorance, it is perhaps not surprising that they are also very similar when all investors are informed about the quality shock, and then their maximization is based on the true quality shock instead of the expected one. The price schedules in the second panel of figure 2 are similar to those in the first panel, except that they are shifted up (and down) in response to a lower (higher) default probability.

Table 1 reports the average debt burden of the government (computed as the expected total amount that the government has to repay in exchange of raising $D = 60$ to investors in case of not defaulting) at the two auctions under our two informational extremes. Here too the UP and DP auctions are very close to each other for each one of the benchmarks. In the appendix, we argue that this result will generally happen whenever the price schedules are fairly flat with respect to the demand shock, as they are in our numerical example.

This finding is consistent with the empirical literature that has looked at the change in average yields before and after the U.S. switched from a DP to a UP protocol. For example, Malvey, Archibald, and Flynn (1995) found that the average spread in the auction yields compared to the WI market (on-the-run treasuries in the secondary market) were not statistically different, though the UP yield spread was slightly smaller.
Table 1: Avg. Debt Burdens and Conditional Variance by n

<table>
<thead>
<tr>
<th># of Informed</th>
<th>UP</th>
<th>DP</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 0 avg.</td>
<td>96.7</td>
<td>96.8</td>
</tr>
<tr>
<td>n = 0 var(·)</td>
<td>7.92</td>
<td>0.06</td>
</tr>
<tr>
<td>n = 1 avg.</td>
<td>101.3</td>
<td>101.4</td>
</tr>
<tr>
<td>n = 1 var(·</td>
<td>θ)</td>
<td>[30.04, 0.92]</td>
</tr>
</tbody>
</table>

DP auctions do have an important advantage relative to UP auctions under symmetric ignorance and, conditional on θ, under symmetric information. In figure 2 we also plotted the average price paid as well as the marginal price for the symmetric ignorance case. The average price paid conditional on η is the marginal price in a UP auction, but not in a DP auction. Here, since transactions occur at the accepted bid price, the bulk of these transactions occur at the highest possible price, since the revenue at this highest price will equal \((1 - η) * D\) from market clearing at the highest price. Hence, the average price paid at a DP auction varies much less than that at a UP auction. As a result, the variance of the debt burden is much smaller. For example, in the case when \(n = 0\), the variance of the debt burden was 0.06 under DP vs. 7.92 under UP. From the table, we can see that a similar result holds if we compute the variance conditional on θ when \(n = 1\).\(^{12}\)

### 6.2 Asymmetric Information with UP Auctions

We now turn to examining what happens in our numerical example when we shrink the number of informed from \(n = 1\) to \(n = 0\) in a UP auction. The results are shown in figure 3. At first, lowering \(n\) has no impact because the debt overhang constraint does not bind and we have perfect replication, as long as \(n > η_M = 0.3\) (see Proposition 3). However, when \(n\) falls below \(η_M\), the binding debt overhang constraint forces the price schedules to overlap. Since the constraint first binds when \(n = η_M\), the initial overlap is such that states \((g, η_M)\) and \((b, 0)\) share a common price. As \(n\) falls further, the bid-overhang constraint binds at progressively lower values of the demand shock on the high quality schedule,

\(^{12}\)The debt burden is higher in the \(θ = b\) state and this raises the conditional variance. We do not focus on the unconditional variance because the variation in the debt burden coming from \(θ\) is so large as to swamp this comparison.
and there is a larger overlap region. The reason is that the per-capita demand that is required for the uninformed’s bid to cover the total demand when the demand shock is at its smallest value is lower as the fraction of uninformed investors increases.

Because the initial price at which the price schedules overlap is on the high quality schedule, $\tilde{\kappa}$ has to be close to $\kappa_g$ to prevent the uninformed from reducing their demands too dramatically. In fact, when the short-sale constraint binds at this price for the informed it is easy to see that it must in fact be equal to $\kappa_g$ since the initial binding point occurs when the demands of the uninformed are just sufficient to cover the total supply of $D$ when $\eta = 0$ and $\theta = b$. This means that the set of $\eta$ for the same price holds is higher when $\theta = g$ than when $\theta = b$. We can see this effect in figure 3 by noting that the slope of the high quality schedule is much flatter than that of the low quality schedule when $n$ is fairly high and the demands of the informed are substantial.

The fact that the prices are falling as the gap between $\eta_g$ and $\eta_b$ widens comes from the need to have a greater spread in the total bids of the informed at the two different values of $\theta$. As we can see from the figure, this can even lead to prices that are below those on the uninformed price schedule (i.e. the $n = 0$ equilibrium).

The figure also shows there is more overlap on prices as $n$ shrinks, and that the high and low quality price schedules are converging to the uninformed price schedule for all interior $\eta$. As $n$ shrinks, also should the gap in the $\eta$’s as the informed investors can cover less and less of the variation in bids across the two quality states. As this leads the $\eta$’s varying one-to-one, in the limit $\tilde{\kappa} \rightarrow \kappa^U$ and the prices tend to lie on the $n = 0$ schedule, except at the two extremes of the $\eta$ grid, $\eta = 0$ and $\eta = \eta_M$.

**Remark 5.** It is interesting to compare the UP auction equilibrium to the set of competitive equilibria in our numerical example. Because the auction equilibrium has an additional constraint in terms of the bid-overhang constraint, when $n \leq \eta_M$ the fully revealing price schedules that hold when $n = 1$ and there is perfect replication, are no longer an auction equilibrium. But the fully revealing equilibrium is still a competitive equilibrium in this case. Thus, we have multiplicity in terms of competitive equilibria once $n \leq \eta_M$, and one of the competitive equilibria features fully revealing prices and one does not.
Figure 3: UP Auctions as Information Shrinks

(a) $n > 0.3$. Debt overhang not binding.

(b) Debt Overhang now binding
6.3 Asymmetric Information with DP Auctions

We now turn to DP auctions and examine how equilibrium prices change as we shrink \( n \) from 1 to 0. Some examples are plotted in Figure 4. It is immediately clear that equilibrium prices are very different compared to the UP auction. The first important observation is that the low quality-price schedule associated with \( \theta = b \) is independent of \( n \) if \( n \) is close enough to 1, while the high quality-price schedule associated with \( \theta = g \) is sensitive to \( n \).

The mechanism underpinning this result is tightly linked to the auction protocol. When \( n \) is large, there is a large spread between the high-price schedule and the low-price schedule. Uninformed investors who bid at the high-price schedule thus face a very severe adverse selection problem because their bids are executed at very high prices when the bond quality is low. To avoid this issue, uninformed investors rather do not bid at all on the high-price schedule when \( n \) is large enough. This has two effects. First, the uninformed know that their bids on the low-price schedule will only be accepted when \( \theta = b \). Conditional on buying, they are thus perfectly informed about the quality shock at which they are buying, and they thus choose the same \( \theta = b \) portfolio as informed investors. As informed and uninformed investors bid the same at the the low-quality schedule this schedule is the same as in the fully informed equilibrium as long as uninformed investors do not bid anything on the high-quality schedule. Second, precisely because the uninformed do not participate in the high state, the informed are disproportionately exposed to the government’s default risk in that state. As a result, marginal prices must fall as \( n \) decreases in order to deliver an increase in the risk premium for those fewer informed investors buying in in the high state. Accordingly, the figure shows a fall in prices as \( n \) declines.

This process continues, forcing prices on the high quality schedule to drop until at around \( n = 0.4 \) in our illustration. Once the high-quality schedule is low enough the uninformed investors are less worried about the over price they may have to pay for bids on the high quality schedule when the bonds are low quality, and begin buying on both schedules. At that point, the bids made at high prices on the high quality schedule
mean that there is less extra demand that has to be squeezed out when the quality is low. Hence this shift in the bidding of the uninformed both slows the fall in the high quality schedule and raises the prices on the low quality schedule. However, the adverse selection effect is so strong, that as \( n \) falls, it forces more and more of the high quality schedule below the uninformed price schedule. The adverse selection faced by a larger fraction of uninformed investors lead almost all prices (except the ones for \( \theta = g \) and \( \eta = 0 \)) to fall below those obtained under symmetric ignorance.

As \( n \to 0 \) and the uninformed dominate the market, the likelihood of buying on the high and low quality schedules converges everywhere, except at the bottom of the low quality schedule (at \( \theta = b \) and \( \eta = \eta_M \)), and hence the prices converge almost everywhere to the \( n = 0 \) schedule.

It is important to notice that, almost by definition, the range of prices at which bids are executed in equilibrium is very different across the two auction protocols. With a UP auction, bids are executed at a unique marginal price given the realized state \((\theta, \eta)\). Nevertheless, the range of realized prices can vary quite a bit across states, especially when \( n > \eta_M \) and the price schedules are therefore far apart. In contrast, with a DP auction both the range of realized marginal prices and the range of prices at which investors purchase bonds given the realized state may vary a lot. In general, the range of potential prices at which an uninformed investor’s bids might be executed is large when (i) \( n \) is large enough to generate a substantial spread between the high-quality price schedule and the low-quality price schedule, and (ii) \( n \) is small enough to lower the prices on the high-quality schedule enough to induce the uninformed investors to bid on both schedules. In our example this is the case for values of \( n \) from 0.4 to 0.5. In other words, the range of potential prices reaches its maximum among intermediate levels of information (this is, intermediate levels of \( n \)).
Figure 4: DP Auctions as Information Shrinks

(a) $n = 1$

(b) Small fall in $n$
6.4 Payoffs and Yields in UP and DP Auctions

We now want to examine the implications of these two protocols for the ex-ante payoffs to the investors and the government and how these vary with the share of informed investors. For the investors this is straightforward: we simply compute their expected utility for different values of $n$. We plot these results for both types of auctions in the top panel of figure 5.

We have not specified a payoff function for the government. The risk-neutral component of their payoff is captured by the the average yield on their bonds. The yield is simply the promised return on the bonds, given by $(1 - P)/P$. The risky component of their payoff is captured by the variation in the average yield conditional on $\theta$, along with the variation induced by $\theta$. The average yield and its average conditional variance for both types of auctions are displayed at the bottom panel of figure 5.\(^{13}\)

Not surprisingly, given that the prices and total bids are very similar across auction protocols under symmetric ignorance ($n = 0$) and symmetric information ($n = 1$), the payoffs to the informed, the uninformed and the yield for the government are also similar across protocols in these symmetric benchmarks. However, the conditional variance of the yield, while small in both cases, is substantially lower under the DP auction protocol. The main reason is that a lion share of the bids are executed at the largest bid price, and then the rest of the variance on prices that come from demand shocks is relatively residual. When we turn to asymmetric information, the payoffs and yields differ widely across protocols.

In the UP auction, the payoff to being uninformed is almost invariant to the degree of information in the market, while the payoff to being informed is monotonically declining in $n$ until it hits the point of complete replication at $n = \eta_M$. From then on, there are no information rents and expected utility is constant for all investors as $n$ increases to 1. The yield on the government debt rises monotonically up until the point of complete replication, and is constant thereafter as the equilibrium becomes invariant with respect to $n$. In

\(^{13}\)Because we have made the default probabilities so different in order to allow for perfect replication in the UP auction, the differences across quality shocks swamp the conditional. However, this would not be true when these differences were smaller. For this reason, we have chosen to focus on the behavior of the average conditional variance.
contrast the average conditional variance shows a strong non-monotonicity. As we depart from the symmetric ignorance case there is more and more dispersion on prices both in the good and bad price schedules, which makes the average of that variance strongly increase with $n$. As prices converge to the perfect replication schedules, the conditional variances decline and hence so does the average value.

**Remark 6.** If default probabilities are closer together, the uninformed investor will not be able to replicate the total bids of the informed investor. This occurs for two reasons: (1) the short-sale constraint will bind as the total bids by the informed at $(b, 0)$ become less than those at $(g, \eta_M)$,
and (2) because the price schedule when \( \theta = g \) and \( \theta = b \) will overlap even at \( n = 1 \). In this case the gains to being informed will stay positive (but small) for values of \( n > n_M \) and the price schedules will vary with \( n \) even for values close to 1. These statements depend critically on the distribution of \( \eta \) not being degenerate, since in that case, perfect replication is always possible.

In sharp contrast, both the payoff to the informed, and the yield are hump-shaped in the DP protocol. When the fraction of informed investors is low, the adverse selection effect discourages participation by the uninformed and depresses prices both in good states (because the uninformed participate less) and in bad states (because the informed bid less). Because this effect is initially stronger the larger the spread between price schedules, the overall impact on yields is hump-shaped and yields reach their maximum at intermediate levels of \( n \).

Once the fraction of informed is large enough, the uninformed no longer participate in the high state, the informed earn rents only because they participate in both states. These high-state information rents are gradually competed away as the share of informed investors increases. This cannibalization effect raises prices in the good state and reduces the yield as \( n \) approaches 1.

The average conditional variance of the yield also displays non-monotonicity, but much lower levels than in UP auctions. The source of non-monotonicity follows the same logic, as we depart from the symmetric ignorance benchmark convergence towards the other benchmark pushes more dispersion in prices for a particular state. In the DP auction, however, this dispersion only happens in the bad schedule, not in the good one, which reduces the average variance over the whole range of \( n \).

**Remark 7.** In contrast to the UP auctions, in DP auctions the gains from being informed are always positive for all values of \( n \), and even when \( \eta \) is degenerate. This is true even when the price schedules do not overlap and completely reveal the state.

The behavior of the yield and the conditional variance present an interesting risk-return trade-off for the government. While the yield on debt is higher for DP auctions when \( n \) is interior and there is asymmetric information, the conditional variance is always smaller. The average conditional variance of the yield is also hump-shaped in DP
auctions, however it is systematically below that with UP auctions.

7 Model with Information Acquisition

We now endogenize the share of informed investors by allowing for information acquisition. All investors are initially uninformed. After learning whether they will make it to the auction, investors can learn the true value of $\theta$ by paying a utility cost of $K$. An investor will choose to become informed so long as the differential benefit of doing so justifies the cost. Otherwise, everyone will choose to be uninformed. Similarly, an investor will choose to stay uninformed if the benefit from doing so is weakly better than becoming informed, otherwise everyone will be informed. Denote by $V^U$ the expected utility of an uninformed investor, and denote by $V^I$ the expected utility of an informed investor. Given that equilibrium prices are a function of the share of informed $n$, we write $V^I(n)$ and $V^U(n)$ to highlight this dependence. The optimality conditions determine the equilibrium level of $n$ thus are

\[
V^I(n) - K \geq V^U(n) \text{ if } n > 0
\]

\[
V^I(n) - K \leq V^U(n) \text{ if } n < 1.
\]

Both of these equations must hold simultaneously in an interior equilibrium in which $n \in (0, 1)$. This requires that both conditions hold with equality. We are now ready to define an equilibrium with information acquisition.

Definition 4. For both the UP and DP auctions, an equilibrium of the model with endogenous information acquisition consists of the measure of informed traders $n^*$, a price schedule $P(s)$, a bid schedule for the uninformed $B^U(s)$, and a bid schedule for the informed $B^I(s, \theta)$. The bid schedules must be solutions to the investors’ problems given $P$ and satisfy no short-selling constraints. The bids and price schedules must satisfy deb-overhang constraints and market clearing for all $(\theta, \eta)$, and $n^*$ must satisfy the information acquisition criterion in (13) and (14).

Utility gap: Naturally, the incentives to acquire information are determined by the utility gap $V^I(n) - V^U(n)$. Figure 6 plots the utility gap for both auction protocols using
our numerical example. There are marked differences: the utility gap is strictly decreasing in the UP auction (panel a), but it is hump-shaped in the DP auction (panel b).

This has important consequences for equilibrium information acquisition. In particular, there is a unique equilibrium in the UP auction in which the level of information acquisition (i.e. \( n^* \)) is decreasing in the utility cost of information \( K \). In the DP auction, instead, it is easy to construct examples in which there are multiple equilibria for the same parameterization of our model. These multiple equilibria will include: (i) a stable equilibrium in which there is no information acquisition, (ii) an unstable equilibrium in which there is a small amount of information acquisition, and (iii) a stable equilibrium in which there is a large amount of information acquisition.

Figure 6: Equilibrium with Information Acquisition

(a) UP Auction
(b) DP Auction

Notice that the incentives to acquire information are larger in DP auctions for all levels of \( n \). The differential gains from acquiring information are almost identical under symmetric ignorance (\( n = 0 \)). To see this recall that the price and bid schedules under symmetric ignorance are very similar in UP and DP auctions, hence the utility of the uninformed investors are also very similar. The incentives to become informed in this equilibrium comes form adjusting bids to the equilibrium prices more accurately once observing the realized probability of default. This is the reason the gains from becoming informed is almost the same (0.026 in the figures) initially. For a slightly positive fraction of informed investors, however, this differential gains from information behave very dif-
ferently across the two protocols, increasing for DP auctions, decreasing in UP auctions, reaching zero in UP auctions (for \( n \geq \eta_M \)) and being always positive for DP auctions (as uninformed investors can never replicate the informed portfolio).

This comparison has striking implications for the existence of asymmetric information under these two auction protocols. When the cost of information acquisition \( K \) is large (above 0.046 in the numerical illustration), only symmetric ignorance is feasible in both auctions. When \( K \) is intermediate (higher than 0.026 but lower than 0.046 in the numerical illustration) symmetric ignorance is also an equilibrium under both protocols, but for the DP auction there is also an equilibrium with asymmetric information. When \( K \) is small (below 0.026) only equilibria with asymmetric information is feasible in both auctions but with the equilibrium fraction of informed agents lower for the UP auction. Finally, when \( K \) is very small (below 0.005) only asymmetric information is sustainable in UP auctions and only symmetric information is sustainable in DP auctions.

### 7.1 Incorporating Dealers

The participants at sovereign debt auctions typically include two fundamental types, those who are solely bidding on their own, and dealers who are both bidding on their own and executing orders for other investors. Because they are generally taking very large positions, these dealers have very strong incentives to acquire information, hence it is natural to think of them as being informed about the quality shock \( \theta \). It is also natural to think of these dealers as having information about the size of the demand shock coming through their order flow. We can think of this information being akin observing a noisy signal about \( \eta \).

One simple way to incorporate this into the model would be having three different types of investors, our original uninformed and informed investors, and a third set called dealers. We could think of the dealers as always participating in the auction, as always being informed about the quality shock \( \theta \) (because of very low information cost, for example) and as receiving a signal about \( \eta \) coming from their order flow. A particularly tractable way to incorporate this signal would be to partition the set of possible \( \eta \)’s, \([0, \eta_M]\)
into two subsets, \([0, \eta_M/2]\) and \([\eta_M/2, \eta_M]\), and assume that this signal informs the dealers as to which partition the realized \(\eta\) is in.

Just as with the quality shock, the dealers would find it optimal to only bid on the prices coming from the relevant partition. Thus, their bids would be responding to the information that they see in their order flow. By increasing the degree of overlap in the partitions, or increasing the number of partitions and shrinking their individual size, we can adjust the model to the amount of information we want to allow the dealers to have.

It is also straightforward to endow the dealer with market power. Assume there is a single large dealer (with a larger \(W\)) with perfect information about the quality shock \(\theta\) and who can perfectly infer \(\eta\) from order flows, then knowing the state \([\theta, \eta]\). The rest of investors are relatively small (price-takers) as modeled above. As before, investors take as given the bids of other investors’ and also the bid of the dealer that determine the market clearing price in each state, choosing their own bids for each possible marginal price. The dealer, however, chooses his bid considering that the residual supply is given by the available bonds after executing the investors’ bids and then that it has the possibility of affecting the equilibrium price. In other words, the investors face a flat residual supply while the dealer does not. In equilibrium these strategies and equilibrium prices are determined jointly.

8 Discussion of Relationship to Literature

Our work is motivated by the complex dynamics displayed by the prices at which governments sell their bonds in auctions, particularly in emerging economies. We have already discussed the contribution of our model to the sovereign debt literature that focuses on understanding and measuring the impact of these dynamics on the welfare of countries. In this section we discuss the relationship between our work and other several important topics and approaches to the question.

The first concerns the foundations of general equilibrium theory (GE) and the question of “where do prices come from?” The second concerns the question of “where the information encoded in prices gets aggregated?”. The third is about auction theory when
there are a large number of bidders for a perfectly divisible good with uncertain common value, and the specific empirical application to the auction of sovereign bonds.

With respect to the question “where do prices come from?,” the price vector in GE is an endogenous object that is not chosen by anyone, yet determined by the accumulated actions of individuals who cannot affect prices. To get around the conundrum, Walras made up his fictional “auctioneer” that matches total supply and total demand in a market of perfect competition (perfect information and no transaction costs). But this has long been considered a thought experiment that did not adequately address the issue.\textsuperscript{14}

One response to the price problem has been the market games literature, which seeks to provide a fuller description of the environment and in which all endogenous objects are selected by the agents (including prices) based upon noncooperative game theory.\textsuperscript{15} Examples of this market game approach include Rubinstein and Wolinsky (1985)’s sequential bargaining model in which buyers and sellers are paired up under complete information each period.\textsuperscript{16}

This problem is more severe when the prices are simultaneously clearing the market and aggregating information as in Lucas (1972), Radner (1979) and Grossman and Stiglitz (1980). This leads to the complementary question “where does the information in prices come from?” as agents “need to know” both the price function and the realized price in order to make their inferences and determine their individual demands.\textsuperscript{17}

Another layer of complication is “where information comes from?” As Grossman and Stiglitz (1976) pointed out, agents have no incentive to look at their private information since all of the information is already encapsulated in the price. As their quantity choice do not reflect their private information, then whose information gets aggregated into prices? This problem manifests as a nonexistence problem if acquiring the infor-

\textsuperscript{14}See Hahn (1989) for a discussion.
\textsuperscript{15}See Gale (2000) for a survey of this literature and for a discussion of the alternative cooperative approach.
\textsuperscript{16}Gale (1987) shows that these sequential bargaining models converge to a common price equilibrium as the number of agents gets large.
\textsuperscript{17}Dubey, Geanakoplos, and Shubik (1987) consider the Nash equilibrium of a sequential trading game with incomplete information where traders make quantity offers to buy and sell and the price is determined by the ratio of the total buy versus sell offers. Here information revelation occurs largely in one-step through the vector of different prices for the different goods.
information is costly; the Grossman-Stiglitz paradox. If information is costly and prices are fully revealing, no individual wants to acquire information. However, if no agent gathers information, prices cannot be fully revealing. Fully revealing information prices are logically impossible. Grossman and Stiglitz (1980) added a second source of noise to prevent the price system being invertible.\(^{18}\)

But even if it exists a fully revealing equilibrium, there is an implementability problem, as it may not be possible to find a trading mechanism that induces it. Jackson (1991) proposes a resolution when the number of agents is finite and there is no price-taking, as they internalize that the extent of information in prices depends upon the demand schedule they submit. Kyle (1989) study proposes a non-competitive rational expectations model in which agents submit limit orders (demand schedules, as in our case) taking into account their effect on the equilibrium price. Golosov, Lorenzoni, and Tsyvinski (2014) propose a decentralized approach which features a sequence of bilateral meetings with take-it-or-leave-it offers.\(^{19}\) Finally, Vives (2014) and Gaballo and Ordonez (2017) propose settings with large centralized markets in which the valuation of each trader has both common and private value components, and the costly signal bundles information about these two components, such that prices can be fully revealing and yet there are incentives to acquire information.

Our paper speaks to both of these problematic aspects of GE by using the structure of an auction to answer where prices come from and by obtaining the conditions for informational gains in two different auction protocols to answer how information gets into prices. In particular, our model features a specific order of moves. First, investors submit their bids (where each bid is a price-quantity pair). Second, a specific protocol is used to select the bids which are accepted and the prices at which they are executed. Information revelation occurs after the marginal price at the auction is revealed. This information revelation may be complete, as in REE. A related paper which takes a similar

\(^{18}\)The existence of equilibria when the shocks are continuous and hence states can have the same price is well known to be problematic, see Allen and Jordan (1998) for a survey of the existence literature. The combination of CARA preferences and jointly normal shocks was key to the construction of an equilibrium in Grossman and Stiglitz (1980), though recently Breon-Drish (2015) has developed a characterization that drops joint normality.

\(^{19}\)A related contribution is Albagli, Hellwig, and Tsyvinski (2014) who develop a dynamic REE with dispersed information in which information enters nonlinearly into prices.
auction-based approach to micro found REE is Milgrom (1981). He considers, however, an auction in which bidders are restricted in the units to buy and where the price is not clouded by a demand (or supply) shocks. Our paper relax both of these aspects.

Our paper also contributes to the auction literature on multi-unit auctions, in particular as it is applied to goods such as sovereign bonds. The backbone of the auction literature is based on the selling of a single object, either to bidders with independent private values\(^\text{20}\) or to bidders with correlated values,\(^\text{21}\) studying the strategic considerations when choosing the bid price. To capture goods such as treasury bonds, however, the literature extended these models to multi-unit auctions, facing the challenge of solving an equilibrium that involves bidders with a double dimensional strategic problem when submitting a combination of quantities and prices.\(^\text{22}\)

Recent work tackles these questions from an empirical perspective. Hortaçsu and McAdams (2010) develop a model, based on Wilson (1979), of a multi-unit discriminatory-price auction to a finite set of potential risk-neutral bidders with symmetric and independent private values, using data from Turkish Treasury auctions to estimate those bidders’ private values. In their model the symmetric equilibrium bidding functions depend on how an individual bid changes the probability distribution of the market clearing price, a complicated object to characterize analytically. Given this theoretical difficulty, they construct a non-parametric consistent estimator of the distribution exploiting a resampling technique.\(^\text{23}\)

The key difficulty the empirical literature confronts is that “common value” and “independent value” auctions are (nearly) observationally equivalent even when one assumes risk-neutrality.\(^\text{24}\) This is because it is hard to distinguish the extent to which bidders ad-

\(^{20}\)See Vickrey (1962), Harris and Raviv (1981), Myerson (1979) and Maskin and Riley (1985).


\(^{23}\)Kastl (2011) extended Wilson model, which is based on continuous and differentiable functions, to more realistic discrete-step functions, showing that in such case only upper and lower bounds on private valuations can be identified, which he does by using the previously methodology on Czech bills auctions.

\(^{24}\)Laffont and Vuong (1996) argue that common value models are observationally equivalent to an affiliated private value model unless there are exogenous variations that allow for identification. Hortaçsu and Kastl (2012) proposes an identification based on dealers observations of their customers bids using discriminatory Canadian treasury auctions, while Hortaçsu, Kastl, and Zhang (2017) use uniform U.S. treasury auctions.
just their bids to take advantage of market power vs. inferences they make from their bid determining the marginal price (the so-called “shading factor”). Risk aversion, which we deem necessary to understand may countries’ auction data, would only complicate the already complicated identification further. Our assumption of price-taking offers a potentially useful empirical as well as theoretical simplification relative to the alternative.

Several differences separate our modeling assumptions from this body of work. First, and most important, our model is based on the presumption that bidders’ valuations of the auctioned treasury bond are perfectly correlated (common value) instead of independent (private value). Laffont and Vuong (1996) argue that common value models are observationally equivalent to an affiliated private value model unless there are exogenous variations that allow for identification. Hortac¸su and Kastl (2012) use Canadian treasury auctions to test whether common values or private values are a better representation of the motivations to buy treasury bonds. In Canada, some bidders (dealers) are allowed to observe the bonds of another set of bidders (costumers) when preparing their own bids. In a private value auction, observing the bid of a costumer only gives information about the competition the dealer faces (and then the probability of winning the auction) but not about the fundamental value of the bond. In this case the dealer would not revise the bid if this was higher than the observed competing bid. In a common value price auction, however, observing a costumer’s bid induce learning not only of competition but also of the fundamental, leading to a revision of the intended bid also when the bid was higher that the observed competing bid. Since Canadian auctions are discriminatory, this testable implication is not as straightforward, but they propose a correction, concluding that there is no evidence in their data that dealers learn about fundamentals from costumers. This is not prima facie evidence that dealers follow private values, but instead that they may have superior information than costumers about the common value.25

The second relevant difference is that we assume investors are risk averse, not risk neutral. Following Wilson (1979), this departure is indeed relevant for the interpretation of the shading factor. The third difference is our modeling of several correlated shocks,

25With a similar methodology applied to uniform-price auctions of U.S. Treasury bills, Hortac¸su, Kastl, and Zhang (2017) estimate that the informational advantage of primary dealers leads them to higher yield bids as a response to a larger ability to bid-shade their bids.
departing heavily from the assumptions of independent realizations across bidders. The quality shock, the demand shocks and the signal that all informed investors receive are perfectly correlated, which implies that bid shading only happens for uninformed investors in response to the possibility of adverse selection and not to competitive forces.

Closer to our setting, Boyarchenko, Lucca, and Veldkamp (2017) study an auction environment with risk averse investors that are asymmetrically informed about the common value of a bond. They consider some investors have superior information and have market power, calibrate the model to U.S. Treasury auctions and show that information sharing across investors increase government revenues as investors are willing to invest more as they become better informed. By focusing on the assumption that investors are price-takers we focus on the effect of asymmetric information on prices instead of on the effect of strategic considerations on prices.

9 Conclusion

We develop a rich model of sovereign debt auctions which features an (implicit) public shock, a quality shock about which there can be heterogeneous information, and a correlated private shock which determines auction participation. Our numerical example illustrates how our model can speak to the kind of data we see in sovereign debt auctions like that for Mexican bonds in figure 1. During crisis periods, when the range of potential default probabilities increase, the model generates a sharp rise and highly volatile interest rates, as those we see during Mexican “Tequila Crisis” of 1995. When the level of a country’s indebtedness increase there is a substantial pressure for information acquisition and information asymmetry (specially under discriminatory-price auctions), decreasing participation of uninformed investors under the threat of adverse selection and increasing interest rates even when the quality shock is good. This reduction in participation is also reminiscent of the decline in bids and failed auctions that we commonly observe during crises. At the same time, under either symmetric information or symmetric ignorance when there is little heterogeneity, the additional risk premia associated with demand shocks is small. This illustrates how the model can also accommodate the relatively
low volatility of the Mexican Cetes interest rates in recent years.

Our model also provides interesting insights into the impact of different auction protocols on the extent of information acquisition. There is a trade-off between the two protocols, with both behaving similarly in terms of payoffs and yields when information is symmetric, though the discriminatory-price auction does offer lower variation in the yields in response to demand shocks. These symmetric cases can be thought of as occurring during tranquil periods when there is no information to be obtained about the government’s future likelihood of default, or during eventful periods in which this information is public. However, during turbulent periods in which there are large incentives to obtain information, the adverse selection effect in discriminatory-price auctions can lead to a wide dispersion in realized auction prices and higher average yields as uninformed investors are reluctant to participate. These results contribute to the wide discussion, dating back at least to Friedman (1960), of whether sovereign debt auctions should be conducted with a uniform-price or a price-discriminating protocol.26 Our results suggest that the benefit of choosing a discriminatory or a uniform price auction depends on the extent of the incentives to acquire private information.

In a follow-up paper (Cole, Neuhann, and Ordonez (2016)) we examine the implications of discriminatory-price auctions within a two-country setting. We use the insights developed here to discuss spillovers of information across countries and the role of secondary markets. We show that the sources of complementarities inherent to discriminatory-price auctions extend from cross-states to cross-bonds and make debt crises contagious even in the absence of other linkages.

The model that we develop is applicable to a number of other important circumstances, including auctions of liquidity infusion by central banks, electricity, emission permits, gas, oil, and mineral rights. The key requirement is that the auction involves a ”thick” enough market for a homogenous divisible good of uncertain quality so as to make the price-taking assumption a close approximation to reality. Our model also pro-

26Friedman proposed (pp 64-65) that the U.S. Treasury abandons its previous price-discriminating practice and make all awards at the stopout price instead of at differing prices down through that price. The U.S. Treasury finally adopted this uniform-price protocol for all auctions of 2-year and 5-year notes on September 3, 1992. An excellent summary of this discussion is Chari and Weber (1992). Earlier discussions about Friedman’s proposal include Goldstein (1962), Friedman (1963), Rieber (1964) and Friedman (1964).
vides a potential mechanism to micro found competitive equilibria for the case of the uniform-price protocol and to break the circularity inherent in having prices and quantities determined simultaneously.

References


10 Appendix

11 Almost Equal Revenue

In the two auctions, under symmetric ignorance and symmetric information, we get almost the same average debt burdens. Here we seek to understand why that happens. To do so, consider a fine grid on the \(\eta\)'s indexed by \(j = 1, ..., J\). Define the f.o.c. kernel for a bond \(B_i\) in state \(j\) (where \(j \geq i\)) under the UP auction as

\[
X_{ij}^{U} = \frac{(1 - \kappa)[1-P_j]}{W - \frac{D}{1-\eta_j} + \frac{D}{1-\eta_j} \frac{1}{P_j}} - \frac{\kappa U P_j}{W - \frac{D}{1-\eta_j}}.
\]

Since \(X_{ij}^{U}\) does not depend upon \(i\), it quickly follows that

\[
X_{ij}^{U} = X_{j}^{U} = \frac{(1 - \kappa)[1-P_j]}{W - \frac{D}{1-\eta_j} + \frac{D}{1-\eta_j} \frac{1}{P_j}} - \frac{\kappa U P_j}{W - \frac{D}{1-\eta_j}}.
\]

Since \(\eta\) is distributed uniformly, the f.o.c. for a bond \(B_i\) can be expressed as

\[
\sum_{j=i}^{J} X_{ij}^{U} = 0 \text{ for all } i,
\]

it follows that

\[
X_{j}^{U} = 0 \text{ for all } j.
\]

Define the f.o.c. kernel for a bond \(B_i\) in state \(j\) (where \(j \geq i\)) under the DP auction as

\[
X_{ij}^{D} = \frac{(1 - \kappa)[1-P_i]}{W - \frac{D}{1-\eta_j} + \frac{D}{1-\eta_j} \frac{1}{P_j}} - \frac{\kappa U P_i}{W - \frac{D}{1-\eta_j}},
\]

where \(\tilde{P}_j\) is the average price which satisfies the condition

\[
\tilde{P}_j \ast B_{R,j} = \frac{D}{1-\eta_j},
\]
where $B_{R,j}$ is the number of risky bonds sold in state $\eta_j$. This average price must be equal to the marginal price when $j = 1$, will decline more slowly than the marginal price as $j$ increases. The first-order condition for bond $B_i$ can then be expressed as
\[
\sum_{j=i}^{J} X_{ij}^{DP} = 0 \text{ for all } i.
\]

Note that if $P_j$ does not decline very much so $P_1$ is close to $P_J$, then $\bar{P}_j \simeq P_j \simeq P_j'$. In this case, the condition becomes very close to that in the UP auction, and
\[
X_{ij}^{DP} \simeq 0.
\]

In figure 2 we compare prices in the symmetric ignorance case with the ”average price” in the DP auction. What we see is that while the marginal prices are fairly flat, with the DP price schedule being flatter than the UP schedule, the ”average price” paid schedule in a DP auction is even flatter. Given this, it is unsurprising in light of the above discussion that the total debt burden is also close, and that the variance of the debt burden is much lower under the DP protocol.